ON QUATERNIONS AND MONOPOLES

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Abstract. It is shown that the quaternionic Hilbert space formulation of quantum mechanics allows a quantization, based on a generalized system of imprimitivity, that leads to a description of the motion of a quantum particle in the field of a magnetic monopole. The corresponding Hamilton operator is linked to the theory of projective representations in the weakened form proposed by Adler.

I. INTRODUCTION

Symmetries are one of the most powerful tools in theoretical physics. And yet there are few, if any, exact symmetries in Nature. Thirty years ago Hans Ekstein[1] addressed this problem by introducing the concept of "presymmetry"– a pre–dynamical symmetry, that is being broken by dynamics and yet is evidenced in the algebra structure. Adler[2] introduced the concept of a "weak projective representation "(WPR) and analyzed it within the framework of quaternionic quantum mechanics (see also [3 4] for the epistemological controversy which arose around this concept). In a recent note Adler and Emch[5] revisited the basic concepts of strong and weak projective representations from the point of view of Wigner's theorem [6] and the axiomatic formulation of quaternionic quantum mechanics extensively analyzed by one of us (GGE) more than thirty years ago [7]. Almost concommitantly, more than twenty years ago, following the original ideas of

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Ekstein[1], one of us (AZJ) introduced the concept of a generalized imprimitivity system (GIS) – a concept which involves operator–valued multiplier as in WPR. In [8] the Stone – von Neumann theorem was enhanced so as to also apply to GIS's, and in [9], as an illustration, a GIS corresponding to a charged quantum particle in the field of Dirac's magnetic monopole was explicitly constructed. In the present paper these approaches are brought together, and we show that WPR's arise naturally from GIS's and that they correspond to symmetries that are only partially broken, with a remaining presymmetry (in the sense of Ekstein) holding only for an Abelian subalgebra of the algebra of all observables. We illustrate these concepts on the example of magnetic monopole quantum mechanics; we show that the, heretofore, somewhat mysterious half-spin properties [10] – by the very nature of the Clifford algebra [11] of $(\mathbb{E}^3; -1, -1, -1)$ – naturally involve a quaternionic Hilbert space formulation.

II. MOTIVATING MODEL

Our model describes quantum kinematics and dynamics of a charged particle in the field of a magnetic monopole. The model is realized in a space of square integrable sections of a Hermitian quaternionic line bundle over $\mathbb{R}^3 \setminus \{0\}$. The basic properties and notations relative to the field of quaternions $\mathbb{H}$ and the quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}} = \mathcal{L}^2(\mathbb{R}^3, d^3x; \mathbb{H})$ are reviewed in the Appendix. From a measure–theoretical point of view the Hilbert spaces $\mathcal{L}^2(\mathbb{R}^3, d^3x; \mathbb{H})$ and $\mathcal{L}^2(\mathbb{R}^3 \setminus \{0\}, d^3x; \mathbb{H})$ are naturally isomorphic and we will not make any distinction between them until section IV, where the differential geometric aspects of the construction will be discussed.

• The position operators are defined, as usual, by $[X \ \psi](x) = x \ \psi(x)$; we denote by $\{E(\Delta) | \Delta \subset \mathbb{R}^3\}$ their spectral family (see Appendix). Our model is spherically symmetric, with the rotation generators M given by $M = \epsilon \quad x \, \partial \ - \frac{1}{2}\hat{e}$, where ϵ is the totally antisymmetric tensor with $\epsilon \quad = 1$ for ijk any cyclic permutation of the indices 123 – so that, e.g. $\epsilon \quad a \ b \ = (\mathbf{a} \times \mathbf{b})$ – and where e_1, e_2, e_3 are the three standard quaternion imaginary units.

For every $0 \neq \mathbf{x} \in \mathbb{R}^3$ let $j(\mathbf{x})$ the imaginary unit quaternion

(2.1a) $$j(\mathbf{x}) = \frac{\mathbf{e} \cdot \mathbf{x}}{\|\mathbf{x}\|} \quad .$$

• The linear operator $J = \hat{j}$, i.e.:

(2.1b) $$(J\psi)(\mathbf{x}) = j(\mathbf{x})\psi(\mathbf{x})$$

satisfies the two relations $J^* J = I = JJ^*$ and $J^* = -J$, i.e. is unitary and anti–hermitian; clearly, we also have $J^2 = -I$. Moreover J is invariant under rotations and commutes with the position operators.

For every direction $\mathbf{u} \in S^2 = \{\mathbf{u} \in \mathbb{R}^3 \mid \|\mathbf{u}\| = 1\}$, we construct an anti-hermitian operator $\nabla_{\mathbf{u}}$ given by the formula:

(2.2) $$\nabla_{\mathbf{u}} = \mathbf{u} \cdot \boldsymbol{\partial} + \frac{1}{2} \frac{\mathbf{e} \cdot [\mathbf{u} \times \mathbf{x}]}{\|\mathbf{x}\|^2} \quad .$$

• $\nabla_{\mathbf{u}}$ generates a one-parameter unitary group $\{U_{\mathbf{u}}(s) \mid s \in \mathbb{R}\}$ which satisfies, for all $s \in \mathbb{R}$ and all Borel subsets $\Delta \subset \mathbb{R}^3$:

(2.3) $$U_{\mathbf{u}}(s)\, E(\Delta)\, U_{\mathbf{u}}(-s) = E(\Delta - s\mathbf{u})\,;$$

or, infinitesimally:

$$[\nabla\ , x\ ] = \delta \quad .$$

Thus $\nabla_{\mathbf{u}}$ generates translations in the direction $\mathbf{u}$ of the position variables. Moreover, we have $[M\ , \nabla\ ] = -\epsilon \quad \nabla$, so that $\boldsymbol{\nabla}$ transforms as a vector under rotations.

• The unitary evolution defined by

(2.4) $$U(t) = \exp(-JHt) \text{ where } H = -\frac{1}{2m}\boldsymbol{\nabla}^2 \quad \text{and} \quad \boldsymbol{\nabla}^2 = \sum_{=1}^{3} (\nabla\ )^2$$

gives the evolution equations for the position operator $\mathbf{X}$, namely :

(2.5a) $$X\ = -\frac{J}{m}\nabla$$

and

$$X \; = \frac{1}{2m}\epsilon \quad (X \;\; B \;\; + B \;\; X \;\;) \tag{2.5b}$$

with

$$[B \; \psi](\mathbf{x}) = \frac{1}{2}\frac{x}{\|\mathbf{x}\|^3}\psi(\mathbf{x}) \quad , \tag{2.6}$$

which correspond to the motion of a charged particle in the field of a magnetic monopole.

• The translation generators do not commute:

$$[\nabla \;\; , \nabla \;\;] = -\frac{1}{2}\epsilon \quad \frac{x}{\|\mathbf{x}\|^3} \; J \tag{2.7}$$

which implies that the unitary operators $\{U(\mathbf{a}) \mid \mathbf{a} \in \mathbb{R}^3\}$ defined by $U(s\,\mathbf{u}) = U_{\mathbf{u}}(s)$ for all $s \in \mathbb{R}$ and $\mathbf{u} \in S^2$, i.e. for all $s\,\mathbf{u} \in \mathbb{R}^3$, are only a WPR of the translation group in the sense of Adler.

• The following "splitting" relations are satisfied:

$$0 = [X \; , J] = [\nabla_{\mathbf{u}}, J] = [H, J] \, . \tag{2.8}$$

III. DETAILS OF THE CONSTRUCTION

The canonical quantization is given by the system of imprimitivity where

$$V(\mathbf{a})E(\Delta)V(-\mathbf{a}) = E(\Delta - \mathbf{a}) \quad \text{with} \quad [V(\mathbf{a})\psi](\mathbf{x}) = \psi(\mathbf{x} - \mathbf{a}) \quad , \tag{3.1}$$

where $\{V(\mathbf{a}) \mid a \in \mathbb{R}^3\}$ is a continuous unitary representation with generators ∂ . These generators correspond to covariant derivatives of the flat connection. In presence of an external magnetic field: vector potential enters into the connection form; covariant derivatives cease to commute; parallel transport becomes path dependent; translational symmetry is partially broken; and an operator–valued multiplier corresponding to an integral curvature enters into the group composition formula. In the present paper we want to draw attention to the clarifying role played by the quaternions; we skip therefore any further heuristic motivation of the construction.

We define, for every $\mathbf{a} \in \mathbb{R}^3$ and for all $\mathbf{x} \in \mathbb{R}^3$ not colinear with $\mathbf{a}$

(3.2)
$$w(\mathbf{a};\mathbf{x}) = \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{\|\mathbf{x}\|^2+\mathbf{a}\cdot\mathbf{x}}{\|\mathbf{x}\|\|\mathbf{x}+\mathbf{a}\|}} + j(\mathbf{x}\times\mathbf{a})\sqrt{1-\frac{\|\mathbf{x}\|^2-\mathbf{a}\cdot\mathbf{x}}{\|\mathbf{x}\|\,\|\mathbf{x}+\mathbf{a}\|}}\right),$$

and let $W(\mathbf{a})$ denote the bounded linear operator $\hat{w}(\mathbf{a};\cdot)$, that is

$$(W(\mathbf{a})\psi)(\mathbf{x}) = w(\mathbf{a};\mathbf{x})\psi(\mathbf{x}) \quad \text{a.e.} \tag{3.3}$$

(see Appendix).

It can be verified that:

• $w(\mathbf{a};\mathbf{x})w(\mathbf{a};\mathbf{x}) \;\; = 1$ a.e., and thus $W(\mathbf{a})$ are unitary operators. They commute with the position observables.

• $w(\mathbf{a};\mathbf{x})$ satisfy the cocycle relations

$$w(t\mathbf{a},\mathbf{x}+s\mathbf{a})w(s\mathbf{a},\mathbf{x})w(s\mathbf{a},\mathbf{x}) = w((s+t)\mathbf{a},\mathbf{x}), \quad \text{a.e.}.$$

For every $\mathbf{a} \in \mathbb{R}^3$, define

$$U(\mathbf{a}) = V(\mathbf{a})W(\mathbf{a})$$

and for each $\mathbf{u} \in S^2$ and $s \in \mathbb{R}$, let

$$U_{\mathbf{u}}(s) = U(s\mathbf{u})\,;$$

• $\{U_{\mathbf{u}}(s) | s \in \mathbb{R}\}$ is a continuous unitary group representation of $\mathbb{R}$ whereas $\{U(\mathbf{a}) | a \in \mathbb{R}^3\}$ will only be a weak projective representation – see below.

• By a direct computation one verifies that, for every direction $\mathbf{u}$, the infinitesimal generator $\nabla_{\mathbf{u}}$ of $U_{\mathbf{u}}(t)$ is given by (2.2) .

• $U(\mathbf{a})$ satisfy the imprimitivity relations (2.3); it follows that for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$:

$$U(\mathbf{a})U(\mathbf{b}) = U(\mathbf{a}+\mathbf{b})M(\mathbf{a},\mathbf{b}) \tag{3.4}$$

with $M(\mathbf{a},\mathbf{b})$ commuting with $E(\Delta)$ for all Borel subsets $\Delta \subseteq \mathbb{R}^3$. Thus $M(\mathbf{a},\mathbf{b})$ are of the form $(M(\mathbf{a},\mathbf{b})\psi)(\mathbf{x}) = m(\mathbf{a},\mathbf{b};\mathbf{x})\psi(\mathbf{x})$. In Adler's notation [1] , this reads $M(\mathbf{a},\mathbf{b}) = \int |\mathbf{x} > m(\mathbf{a},\mathbf{b};\mathbf{x}) < \mathbf{x}| \, d^3x$. Upon writing $m(\mathbf{a},\mathbf{b};\mathbf{x})$ in terms of $w(\mathbf{a};\mathbf{x})$ we find:

$$m(\mathbf{a},\mathbf{b};\mathbf{x}) = w(\mathbf{a}+\mathbf{b};\mathbf{x}) \;\; w(\mathbf{a};\mathbf{x}+\mathbf{b})w(\mathbf{b};\mathbf{x}) \in \mathbb{H}. \tag{3.5}$$

In fact, by a direct calculation, we receive:

$$m(\mathbf{a},\mathbf{b};\mathbf{x}) = \exp(J\Phi(\mathbf{a},\mathbf{b};\mathbf{x}))\,, \tag{3.6}$$

where $\Phi(\mathbf{a},\mathbf{b};\mathbf{x})$ is the flux of the monopole magnetic field through the flat triangular surface spanned by the vertices $(\mathbf{x}, \mathbf{x}+\mathbf{a}, \mathbf{x}+\mathbf{a}+\mathbf{b})$. The cocycle formula for $M(\mathbf{a},\mathbf{b})$ expressing associativity of the operator product $(U(\mathbf{a})U(\mathbf{b}))U(\mathbf{c}) = U(\mathbf{a})(U(\mathbf{b})U(\mathbf{c}))$ is then interpreted as stating that the flux through the closed tetrahedron spanned by the edges $(\mathbf{x}, \mathbf{x}+\mathbf{a}, \mathbf{x}+\mathbf{a}+\mathbf{b}, \mathbf{x}+\mathbf{a}+\mathbf{b}+\mathbf{c})$ is an integer multiple of 2π which is automatically satisfied by the magnetic field of the monopole - see (2.6).

IV. DISCUSSION

Our magnetic monopole model is constructed in a quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$, yet it admits a commuting antiunitary involution J and thus reduces, de facto, to a complex Hilbert space model in $\mathcal{H}$. The phenomenon of a "weak projective representation", in the sense implied by Adler, here for the translation group, shows up in both the quaternionic space and in the complex reduction. This is because the "twisted translations" $U(\mathbf{a})$ commute with J. A differential geometric interpretation of the construction is helpful in order to understand at a deeper level what is really going on here. The Hilbert space $\mathcal{H}_{\mathbb{H}} = \mathcal{L}^2(\mathbb{R}^3, d^3x; \mathbb{H})$ can be considered as a Hilbert space of square integrable sections of a trivial Hermitian complex line bundle F over $\mathbb{R}^3\backslash\{\mathbf{0}\}$. Removing the origin results in no measurable theoretic consequences; this removal however does have differential geometric sequels. Our operators $\nabla_{\mathbf{u}}$ define a Hermitian connection in F. The curvature two–form Ω, with values in the Lie algebra $su(2)$ is given by the formula

$$\Omega = -\frac{1}{2}\epsilon \quad \frac{x \ x}{\|\mathbf{x}\|^4}\, dx \ \wedge dx \ . \tag{4.1}$$

The fact that the operator J defined by (2.1) commutes with $\nabla_{\mathbf{u}}$ can be interpreted as stating that the map $\mathbf{x} \to j(\mathbf{x})$ is a parallel section of the bundle of quaternionic right-linear endomorphisms of F. The formula (A.10) defining $\mathcal{H}$ describes, de facto, a construction of a Hermitian complex subbundle F of F which reduces the connection $\boldsymbol{\nabla}$. The complex Hilbert space $\mathcal{H}$ consists of square–integrable sections of the bundle F . Because J is invariant under rotation, it follows that the rotation group acts covariantly on F and unitarily on $\mathcal{H}$ and is a two-valued representation of $SO(3)$ corresponding to spin one–half. At first sight, it might appear somewhat surprising that we can have spin one–half in a Hilbert space of complex, one-component, functions. To answer this puzzle, we note that the bundle F is non–trivial. It admits no continous, nowhere zero, sections – it carries a spin one–half "kink". To see that the bundle is nontrivial we compute the simplest topological invariant, that is its first Chern class. In our case it is the integral of the curvature two–form κ, with now:

$$\kappa = -\frac{1}{2}\epsilon \quad \frac{x}{\|\mathbf{x}\|^3} dx \ \wedge dx \ , \tag{4.2}$$

over the sphere S^2 - the result is 2π which proves that the bundle is non–trivial.

While $\nabla_{\mathbf{u}}$ (and thus H) have a simple explicit form, as globally defined differential operators on a dense domain of differentiable functions in $\mathcal{H}_{\mathbb{H}}$, which is built out of sections of a trivial vector bundle over $\mathbb{R}^3\backslash\{\mathbf{0}\}$ – their restriction to $\mathcal{H}$ cannot be so written; this is due to the fact that $\mathcal{H}$ is defined in terms of sections of a non–trivial subbundle over $\mathbb{R}^3 \setminus \{\mathbf{0}\}$. If we were to force an explicit expression for the covariant derivative in the reduced bundle, a string–like singularity would have to appear – a one–point singularity on each sphere of the constant radius $0 < r \in \mathbb{R}$. Hence the definite advantage of working with the quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$.

Working with singularity–free formulation does not depend by itself on the quaternionic structure – we could use as well a $\mathbb{C}^2$ bundle; nevertheless, the full gauge freedom of the theory is manifest only from within a quaternionic perspective.

Let us, finally, comment upon the relations between the present work and GIS'es studied in Refs.[8 9]. To define a GIS we need an action of a group G on a space X. In the most regular case, G is a Lie group acting differentiably on a manifold X. A GIS is then defined by the relations:

$$U(g)E(\Delta)U(g) \ = E(\Delta g) \tag{4.3a}$$

$$U(g)U(h) = U(gh)M(g,h) \tag{4.3b}$$

$$M(g,h) = \int \ m(g,h;x)dE(x)\,, \tag{4.3c}$$

where $g :\rightarrow U(g)$ is a continuous map from G into unitary operators acting on the Hilbert space $\mathcal{H}$, and $m(g,h;x)$ commute with the spectral measure. In the example discussed in the present paper X is the three–dimensional Euclidean space $\mathbf{E}^3$, G is its translation group, and $m(g,h;x)$ are quaternionic valued. It is seen that a GIS always gives rise to a WPR in the sense of Adler. It is however to be remarked that the very concepts of a GIS (and also of WPR) has little to do with the field over which the Hilbert space is defined. The concept applies to real, complex or quaternionic Hilbert spaces as well.

The idea of a "presymmetry" – that is of a symmetry group which is partially broken by the dynamics, but yet still corresponds to a full symmetry group on an Abelian subalgebra – is quite naturally supported by the GIS framework; in contrast, the a priori mathematically more general concept of WPR leaves open the choice of the sub–algebra necessary to the physical interpretation of the group of (pre–)symmetries; the formulation in terms of GIS seems therefore to help specify physically the choice latent in the WPR formulation.

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APPENDIX

The field $\mathbb{H}$ of the (real) quaternions is obtained upon equiping the 4–dimensional real vector space

(A.1)
$$\mathbb{H} = \{ \, q = \sum_{=0}^{3} q \;\, e \;\; | \, a \;\; \in \mathbb{R} \, \}$$

with the non–commutative multiplication it inherits from

(A.2)
$$e \;\, q = q \, e \quad \forall \, q \in \mathbb{H}; \; e \;\, e \;\; = -\delta \;\; e_0 + \epsilon \;\;\; e \;\; , \qquad i, j, k = 1, 2, 3.$$

$\mathbb{H}$ is equiped with the involution

(A.3)
$$q = \sum^{3} a \;\, e \;\; \to q^* = \sum^{3} a \;\, e^* \quad \text{with} \quad e^* = e \quad \text{and} \quad e^* = -e \quad .$$

Note that $SU(2, \mathbb{C})$ is isomorphic to $\{ \, q \in \mathbb{H} \mid q^* \, q = e_0 \, \}$, with the isomorphism given by the identification

$$e \;\; = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

i.e.

(A.4)
$$e \;\; = I \quad \text{and} \quad e \;\; = -i\sigma$$

where the σ are the three Pauli matrices. For any such quaternion, the map

(A.5)
$$\alpha \;\; : q \in \mathbb{H} \mapsto \omega^* \, q \, \omega \in \mathbb{H}$$

is an automorphism of $\mathbb{H}$, and every automorphism of $\mathbb{H}$ can in fact be implemented in this manner. In particular, if ω is an imaginary unit, i.e.

$$\text{(A.6)} \qquad \omega^* = -\omega \qquad \text{and} \qquad \omega^*\omega = e \quad ,$$

then

$$\text{(A.7)} \qquad \alpha \; [q] = q \quad \text{iff} \quad q \in \mathbb{C} \;\; = \{ \, u\, e \;\; + v\, \omega \mid u, v \in \mathbb{R} \, \} \quad .$$

Note that $\mathbb{C}$ inherits from $\mathbb{H}$, the structure of the field $\mathbb{C}$ of the complex numbers.

We will henceforth use the notations $1 = e$, and $e = (e_1, e_2, e_3)$, and for $x \in \mathbb{R}^3$, $x \cdot e = \sum_{=1}^{3} x \; e$. Note that $q^* q = \|q\|^2$ defines the quaternion norm, and that $(x \cdot e)^*(x \cdot e) = \|x\|^2 = \sum_{=1}^{3} (x \;)^2$.

The Hilbert space $\mathcal{H}_{\mathbb{H}} = \mathcal{L}^2(\mathbb{R}^3, d^3x; \mathbb{H})$ is the space of "functions" $\psi : \mathbb{R}^3 \mapsto \mathbb{H}$, square–integrable with respect to Lebesgue measure d^3x. Its vector space structure is defined with multiplication by scalars written from the right:

$$\text{(A.8)} \qquad [\psi q](x) = \psi(x)\, q \, ,$$

and the scalar product is given by:

$$\text{(A.9)} \qquad (\varphi, \psi) = \int_{\mathbb{R}^3} d^3x \, \varphi(x)^* \, \psi(x) \quad .$$

It is linear in its *second* factor, and skew adjoint; hence $(\varphi q_1, \psi q_2) = q_1^* \, (\varphi, \psi) \, q_2$.

The linear operators $A : \mathcal{H}_{\mathbb{H}} \to \mathcal{H}_{\mathbb{H}}$ are denoted with left action, so that $A(\psi q) = (A\psi) q = A \psi q$. The adjoint is defined, as usual, by $(\varphi, A^* \psi) = (A\varphi, \psi) \; \forall \; \varphi, \psi \in \mathcal{H}_{\mathbb{H}}$.

Let E be the spectral family $[E(\Delta)\psi](x) = \psi(x) \chi_\Delta(x)$ where Δ runs over all Borel subsets of $\mathbb{R}^3$, and χ_Δ is the indicator function of Δ.

We denote by $\hat{e}$ the linear anti-hermitian on $\mathcal{H}_{\mathbb{H}}$ defined by left quaternion multiplication $(\hat{e} \;) \psi(x) = e \; \psi(x)$.

More generally, for each bounded measurable function $f : \mathbb{R}^3 \to \mathbb{H}$ let $\hat{f}$ denote the bounded linear operator on $\mathcal{H}_{\mathbb{H}}$ defined by

$$(\hat{f}\psi)(x) = f(x)\psi(x) \quad \text{a.e.}$$

The (real) commutant of E consists then exactly of the operators of the form $\hat{f}$.

For any unitary and anti–hermitian operator J and any fixed imaginary unit ω, let

$$\mathcal{H} \quad = \{\, \psi \in \mathcal{H}_{\mathbb{H}} \mid J\,\psi = \psi\,\omega \,\} \quad . \tag{A.10}$$

Note that $\mathcal{H}$ inherits from $\mathcal{H}_{\mathbb{H}}$ the structure of a complex Hilbert space over the copy $\mathbb{C}$ – see $(A.7)$ – of the field of complex numbers. Specifically, $\varphi, \psi \in \mathcal{H}$ and $z \in \mathbb{C}$ imply $\varphi + \psi \in \mathcal{H}$, $(\varphi, \psi) \in \mathbb{C}$, and $\psi z \in \mathcal{H}$. Furthermore, for every $\psi \in \mathcal{H}_{\mathbb{H}}$, and every imaginary unit $\tilde{\omega}$ such that $\tilde{\omega}\,\omega = -\omega\,\tilde{\omega}$, there exists a unique pair

$$\psi_1, \psi_2 \in \mathcal{H} \qquad \text{such that} \quad \psi = \psi_1 + \psi_2\,\tilde{\omega} \quad ; \tag{A.11}$$

specifically

$$\psi_1 = \frac{1}{2}(\psi - J\,\psi\,\omega) \quad \text{and} \quad \psi_2 = -\frac{1}{2}(\psi + J\,\psi\,\omega)\,\tilde{\omega} \quad . \tag{A.12}$$

Note that as vectors in $\mathcal{H}_{\mathbb{H}}$, ψ_1 and ψ_2 are mutually orthogonal. Therefore the "splitting" (or "dimension–doubling")

$$\psi \in \mathcal{H}_{\mathbb{H}} \mapsto \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{H} \quad \oplus \mathcal{H} \tag{A.13}$$

is a bijective isometry.

References

1. H. Ekstein, Phys. Rev. **153** (1967), 153; Phys. Rev. **184**, 1315.
2. S.L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford U.P., New York, 1995;S.L. Adler, *Projective Group Representations*, J. Math. Phys. **37** (1996), 2352–2360.

3. G.G. Emch, *Comments on a Recent Paper by S. Adler ...*, J. Math. Phys. **37** (1996), 6582–6585.
4. S.L. Adler, *Response to the Comments by G. Emch ...*, J. Math. Phys. **37** (1996), 6586–6589.
5. S.L. Adler and G.G. Emch, *A Rejoinder on quaternionic projective representations*, J. Math. Phys. **38** (1997), 4758–4762.
6. G. Emch and C. Piron, *Note sur les symétries en théorie quantique*, Helv. Phys. Acta **35** (1962), 542-543; *Symmetry in Quantum Theory*, J. Math. Phys. **4** (1963), 469–473.
7. G.G. Emch, *Mécanique quantique quaternionienne et Relativité restreinte. I.*, Helv. Phys. Acta **36** (1963), 739–769; *II.*, Helv. Phys. Acta **36** (1963), 770–788; *Representations of the Lorentz Group in Quaternionic Quantum Mechanics (presented at the Lorentz Group Symposium, Summer 1964*, Lectures in Theoretical Physics, Vol. VIIa, W.E. Brittin, ed., U.Colorado Press,, Boulder CO, 1964, pp. 1–36.
8. A. Jadczyk, *On a theorem of Mackey, Stone and von Neumann for projective imprimitivity systems*, Techn. Rep. IFT, Uni. Wroclaw, no. **328** (1975).
9. A. Jadczyk, *Magnetic charge quantization and generalized imprimitivity systems*, Int. J. Theor. Phys. **14** (1975), 183–192.
10. A.S. Goldhaber, Phys.Rev. **140** (1965), 1407–1413.
11. I.R. Porteous, *Clifford Algebras and the Classical Groups*, Cambridge Univ. Press, Cambridge, 1995.