Algebras, Symmetries, Spaces *

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Abstract

After discussing several aspects of noncommutative geometry from a rather subjective point of view, algebraic techniques are shown to offer a powerful tool for studying specific manifolds in the realm of commutative geometry, with possible generalization to infinite dimensions.

1 Noncommutative Geometry

1.1 Why quantum space?

A possible conclusion stemming from the (till now unsuccessful) experience with relativistic quantum field theory is that the classical space–time model breaks down at very small distances and it has to be replaced by some kind of a 'quantum space'. Thus, if you 'zoom' several dozen times, you see no space and no time. No smooth manifold structure ... only deadly noncommutative 'algebra foam'. It may seem that

noncommutative geometry is the way.

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1.2 The idea of noncommutative geometry

In noncommutative geometry sets are first replaced by algebras and then forgotten. Formally: If

$$a : M \longrightarrow N \tag{1}$$

is a map between spaces, and if $A(M)$ and $A(N)$ are the respective algebras of functions on these sets, then all the information about the map a is, equivalently, contained in the induced map α between the function algebras

$$(\alpha f)(x) \equiv f(a(x)) \tag{2}$$

which is an algebra homomorphism:

$$\alpha : A(M) \longleftarrow A(N). \tag{3}$$

Notice that the arrow has changed its direction! Thus follows the *Algorithm* of noncommutative geometry:

> Forget about M's, N's, and a's. Reverse the arrows, and play with $A(M)$'s $A(N)$'s α's alone, without assuming them commutative. But: do it so as if M's,N's and a's existed.

1.3 Applications

Applications of the ideas of noncommutative geometry can be found today in several areas:

- Algebraic geometry (still Abelian)
- Super–geometry (Grassmann algebras): Berezin, Leites, Kostant, Manin, Penkov[1]
- "Color" generalizations (arbitrary grading groups): Rittenberg, Marcinek[2]
- Simple models: Spera, Dubois–Violette, Madore, Kerner, Connes, quantum groups literature,
- Space–time out of an algebra: Bannier (see Ref. [5]), Ocneanu[3]

[1] See also Ref. [1, 2, 3]
[2] See [4] and references there
[3] Private communication

1.4 And beyond

One may wonder if

ALGEBRA IS THE ANSWER TO LIFE, UNIVERSE AND EVERYTHING ?

I am not quite sure that this is indeed the case. I was lucky enough to be given a chance of learning from Prof. Rudolph Haag, whom you may know as one of the founders of the algebraic approach to quantum theory. Once, when in Hamburg, I went to Prof. Haag, all excited about a new idea concerning algebraic description of gravity. I was surprised to hear this question: "*... yes, but why an algebra ?*" From this time on I have kept repeating this to myself: *why an algebra ?* And also: *why a foam ?* It was another idea discussed frequently in Hamburg (usually Detlev Buchholz and Klaus Fredenhagen were active parties in these discussions) that, perhaps, at an extreme zoom, at very small distances, space–time *smoothes out* again, and an "essentially free", conformally invariant field theory may be at work at this scaling limit. It is at least interesting to renounce, for a while, the "algebra paradigm", including its current season's overcoat, the 'noncommutative geometry paradigm'. When we look for alternatives, we realize that

Reversing the arrows is not what the tigers like best.

What is then more fundamental, more primitive, than algebras? We may think of orthomodular lattices (von Neumann, Jauch, Piron, ...) or operational logics (Randall, Foulis, ...). One of the important conclusions that one can arrive at, while analyzing foundations of quantum theory using these techniques, is that[4]

quantization is the result of restriction on the set of available states.

Thus:

States *are more primitive than* **algebras**.

Notice that if

$$\alpha : A \longrightarrow B \tag{4}$$

[4]See e.g. [6]

is an algebra homomorphism, then

$$\alpha_\star : B^\star \longrightarrow A^\star , \tag{5}$$

the induced map between states, goes the reverse way. Thus: *arrows get back their original directions.*[5] The generalization from "classical" to "quantum" is now encoded in the convex geometry of the space of states: the set of states is not necessarily a *simplex* and the knowledge of pure states does not longer suffice.[6] Thus: *mixed states* become important[7],[8], and one finds that it is necessary to study *convex and differential geometry of state spaces*[9], including the study of manifolds embedded there[10]. According to this Philosophy algebras should be invoked only when they arise as duals of state spaces.[11] Till now no really satisfactory alternative model based on this principle has been constructed.[12]

2 Studying manifolds by noncommutative algebra techniques – an easy divertimento and illustrative example

Instead of replacing manifolds by algebras, we will consider here *manifolds embedded in algebras and studied by algebra techniques.*

WHY ILLUSTRATIVE ?

- bundle of C*–algebras

[5]Which agrees with "the natural order of things."

[6]See e.g. Ref.[7]

[7]Anyway they are important for OPEN systems; and quantum theory of open systems may even become a necessity if one wants to incorporate equivalence principle.

[8]Note added on February 17, 2001: this paper was written in July 1989. A year later the "Quantum Future" project began, which resulted in EEQT, "Event Enhanced Quantum Theory" [8] , where it was shown that for quantum theory to describe time series of **events**, *open system algorithms* must necessarily be used

[9]in particular the most interesting infinite– dimensional case.

[10]E.g. Phase space can be considered in some cases as a submanifold of the state space, the embedding being implemented via coherent states.

[11]See e.g. Ref.[9]

[12]Notice however the discussion in Piron, Giovannini, Reusse [10, 11, 12], and also the discussion of probabilistic interpretation of the nonlinear Schrödinger equation: Ref. [13, 14]

- Kähler manifolds naturally embedded into projective Hilbert space of quantum states
- for p=q=2 conformal symplectic manifolds
- give rise automatically to a kind of noncommutative geometry... (reserved for future applications)

References to this Section: See [15, 16, 17, 18, 19, 20]
Introduce the following notation:

$$\begin{array}{lcl} V & - & \text{vector space of complex dimension}\, n \\ \langle\, ,\, \rangle & - & \text{scalar product } \equiv \text{nondegenerate sesquilinear form on}\, V \\ B(V) & - & {}^\star\text{–algebra of all linear operators on}\, V \\ \mathcal{S}(V) & - & \text{space of } C^\star\text{–algebra structures on}\, B(V) \end{array} \tag{6}$$

More precisely, $\mathcal{S}(V)$ is defined as follows:

$$\begin{aligned} \mathcal{S}(V) \;\; = \;\; & \{J \in B(V) : J = J^\star = J^{-1}, \text{ and}\, (v_1, v_2)_J \doteq \langle v_1, Jv_2\rangle \\ & \text{is a Hilbert space (thus positive-definite) scalar product on}\, V\} \end{aligned} \tag{7}$$

Notice that each $J \in \mathcal{S}(V)$ defines a $C^\star$ –algebra conjugation $A \mapsto A^J \doteq JA^\star J$, which is the adjoint with respect to the scalar product $(\, ,\,)_J$: $(Av, w)_J = (v, A^J w)_J$.

The manifold $\mathcal{S}(V)$ will be the subject of our study. It carries the following remarkable predicates:

- HOMOGENEOUS
- IRREDUCIBLE
- HERMITIAN
- SYMMETRIC [13]
- COMPLEX
- KÄHLER
- EINSTEIN

[13] We choose the letter $\mathcal{S}(V)$ to remind the fact that $\mathcal{S}(V)$ is isomorphic to the set of all geodesic **symmetries** of $S(V)$.

- NONCOMPACT
- BOUNDED DOMAIN

It is our aim here to take advantage of the fact that $\mathcal{S}(V)$ is realized as a particular submanifold in the algebra $B(V)$, and thus allows study by *algebraic* techniques. In particular we shall study:

- Riemannian (i.e. positive definite) metric g on $\mathcal{S}(V)$
- fundamental form ω
- almost complex structure J
- geodesic transport t
- boundary projection π

2.1 Relation to Twistors

If $\langle \, , \, \rangle$ has signature $(+,-) = (p,q)$, $p+q=n$, then to each p–dimensional $(+)$–subspace $Z \subseteqq V$ one associates the operator $S_Z \in \mathcal{S}(V)$:

$$S_Z = 2E_Z - 1, \tag{8}$$

where E_Z is the orthogonal projection[14] onto Z. Conversely, for each $S \in \mathcal{S}(V)$, the range (=*co–kernel*) subspace of the projection $E_S \doteq \frac{S+I}{2}$:

$$Z_S = \{v \in V : Sv = v\} \tag{9}$$

is maximal (i.e. p–dimensional) positive. $\mathcal{S}(V)$ is a homogeneous space for the unitary group $U(V) \approx U(p,q)$; the natural action can be also described by $S \mapsto USU^\star$, with the isotropy group $U(p) \times U(q)$. Since the central circle group of $U(V)$ acts trivially on $\mathcal{S}(V)$, we get the isomorphism

$$\mathcal{S}(V) \cong \frac{SU(p,q)}{S(U(p) \times U(q))}. \tag{10}$$

Notice that the denominator is the maximal compact subgroup of the numerator.

[14]'Orthogonal' with respect to each one of the two relevant scalar products: the indefinite, and the Hilbert space one obtained by flipping the sign of the complementary subspace

2.2 Tangent Spaces

First of all, the relations $S = S^2$, $S = S^\star$ allow us to identify the space $T^c_S\mathcal{S}(V)$ of complex tangent vectors at $S \in \mathcal{S}(V)$ with operators W such that $WS + SW = 0$. Real tangent vectors ($\in T_S\mathcal{S}(V)$)are characterized by the extra condition $W = W^\star$. The Lie algebra Lie$(U(V))$ can be identified with anti–Hermitian operators $X = -X^\star$ on V. They induce fundamental (real) vector fields on $\mathcal{S}(V)$:

$$\tilde{X} : S \mapsto [X, S] \tag{11}$$

2.3 Riemannian metric

A Kählerian metric g on $\mathcal{S}(V)$ is simply given by[15]

$$g_S(W_1, W_2) = \mathrm{Tr}(W_1 W_2). \tag{12}$$

With this metric $\mathcal{S}(V)$ becomes a symmetric space: each $S_0 \in \mathcal{S}(V)$ is at the same time in $U(V)$, and defines the map

$$\begin{array}{rcl} S_0 : \mathcal{S}(V) & \mapsto & \mathcal{S}(V), \\ S & \mapsto & S_0 S S_0, \\ S_0 & \mapsto & S_0, \\ T_S\mathcal{S}(V) \ni X & \mapsto & S_0 X S_0 = -X \end{array} \tag{13}$$

which is a geodesic symmetry at S_0.

2.4 Almost complex structure

A natural almost complex structure J on $\mathcal{S}(V)$, $J_S : T_S\mathcal{S}(V) \longrightarrow T_S\mathcal{S}(V)$, is given by

$$J_S : W \mapsto J_S W \doteq iSW. \tag{14}$$

Check that J_S maps T_S into itself:
if $X = X^\star$ and $SX + XS = 0$ then $(J_S X)^\star = -iX^\star S^\star = -iXS = iSX$,

$$S(iSX) + (iSX)S = iX + iSXS = iX - iX = 0. \tag{15}$$

[15]The methods apply as well to infinite dimensions, but in infinite dimensions one has to take a special care about existence of trace

Check that $J_S^2 = -1$:

$$J_S J_S X = i^2 S^2 X = -X. \tag{16}$$

We also have

$$g_S(J_S X, J_S Y) = g_S(X, Y). \tag{17}$$

The field J is parallel: $\nabla J = 0$, thus $\mathcal{S}(V)$ is Kählerian.

2.5 Fundamental symplectic form

The symplectic form ω is

$$\omega_S(W_1, W_2) = i\mathrm{Tr}(W_1 S W_2), \tag{18}$$

for W_1, W_2 tangent at S to $\mathcal{S}(V)$. Both g and ω are evidently invariant under the action of $U(V)$. The symplectic form is closed $d\omega = 0$.

2.6 The momentum mapping

For a symplectic manifold $(\mathcal{D}, \omega)$ with a symplectic action of a Lie group G one defines the momentum mapping (Poincaré–Cartan form) as a function $\hat{J} : \mathcal{D} \longrightarrow \mathrm{Lie}(G)^*$ satisfying the condition

$$d(\hat{J}(X)) = i_{\hat{X}}\omega, \tag{19}$$

for all $X \in \mathrm{Lie}(G)$, where for all $s \in \mathcal{D}$, the function $\hat{J}(X)$ is defined by $\hat{J}(X)(s) \doteq \langle J(s), X \rangle$, and $\hat{X}$ is the fundamental vector field associated to X. An explicit knowledge of the momentum mapping is quite useful for a physical interpretation of the geometrical quantities. We can easily compute the momentum map by using the introduced algebraic technique. In our case the momentum mapping is given by a simple formula

$$\hat{J}(X)(S) = 2i\ \mathrm{Tr}(SX), \tag{20}$$

where S is in $\mathcal{S}(V)$ and $X = -X^\star$ is in $\mathrm{Lie}(G)$.

2.7 Geodesic transport formula

To see that the almost complex structure J is covariantly constant under the Levi–Civita connection of g, it is again convenient to use the algebraic machine that provides an easy tool for describing the geodesic parallel transport on $\mathcal{S}(V)$. Given two points $S_1, S_2 \in \mathcal{S}(V)$, the operator $S_1 S_2$ is positive with respect to the p.d. scalar products $(v, w)_{S_i}$, $i = 1, 2$. The operator

$$t_{1,2} \doteq (S_1 S_2)^{\frac{1}{2}} \tag{21}$$

is then unambiguously defined, positive for both scalar products, and an isometry of V; we have $t^\star_{1,2} = {t_{1,2}}^{-1} = t_{2,1}$. Moreover,

$$t_{1,2} S_2 t^\star_{1,2} = S_1 \tag{22}$$

and $t_{1,2}$ maps the p–plane V_{S_2} onto V_{S_1}. The most interesting property of $t_{1,2}$ is that when applied to tangent vectors to $\mathcal{S}(V)$ at S_2 it maps them into the tangent vectors at S_1 obtained by parallel transport along the unique geodesic connecting the two points. To see this one uses the fact that geodesics on $\mathcal{S}(V)$ are trajectories of one–parameter subgroups of $U(V)$. The transport operators t preserve the almost complex structure J on $\mathcal{S}(V)$.

2.8 Boundary map and Cayley transform

Assume $p = q$. The Shilov boundary $\hat{S}(V)$ of $\mathcal{S}(V)$ is defined as the minimal set on which bounded holomorphic functions attain their maximum. It consists here of *isotropic* p–planes. Let us fix one such plane denoted ∞. Each S, being in particular a symmetry in $\mathcal{S}(V)$, reflects ∞ onto another isotropic n–plane

$$\Pi_\infty(S) \doteq S\infty. \tag{23}$$

The map $\Pi_\infty : \mathcal{S}(V) \longrightarrow \hat{S}(V)$ is equivariant with respect to the stability group at ∞. If now an origin O is fixed in $\mathcal{S}(V)$, its image under Π_∞ is called the antipode of ∞ or the origin o of $\hat{S}(V)$. The Shilov boundary $\hat{S}(V)$ carries a natural (flat) Lorentzian conformal structure. It is a homogeneous space not only for $U(V)$ but also for the stability group of each point $S \in \mathcal{S}(V)$.

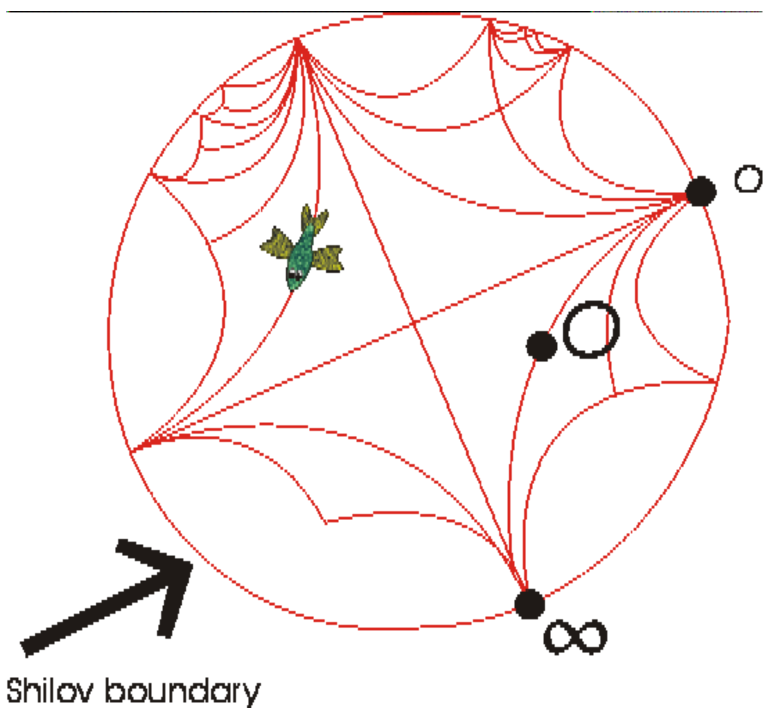

Figure 1:

2.9 C* – algebra bundle

Each point $S \in \mathcal{S}(V)$ determines a Hilbert space scalar product

$$\mathcal{S}(V) \ni S \mapsto (u, v)_S \doteq \langle u, Sv \rangle \tag{24}$$

to which there correspond the "star":

$$A \mapsto A^S \doteq SA^\star S. \tag{25}$$

In this way we produce a bundle of $C^\star$–algebras over $\mathcal{S}(V)$; the fiber over S consists of the algebra $L(V)$ endowed with the "star" conjugation $A \mapsto A^S$. One should notice that the fibers identify here as algebras accidentally owing to the homogeneity of the geometry.

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