

Topics in Quantum Dynamics

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1 Introduction

1.1 The Two Kinds of Evolution

In these lectures I will discuss the two kinds of evolution of quantum systems:

- CONTINUOUS evolution of closed systems
- STOCHASTIC evolution of open systems.

The first type concerns evolution of *closed, isolated*¹ quantum systems that evolve under the action of prescribed external forces, but are not disturbed by observations, and are not coupled thermodynamically or in some other irreversible way to the environment. This evolution is governed by the Schrödinger equation and is also known as a *unitary* or, more generally, as an *automorphic* evolution. In contrast to this idealized case (only approximately valid, when irreversible effects can be neglected), quantum theory is also concerned with a different kind of change of state. It was first formulated by J. von Neumann (cf. [30, Ch. V. 1]) and is known as von Neumann – Lüders *projection postulate*. It tells us roughly this: if some quantum mechanical observable is being measured, then - as a consequence of this measurement - the actual state of the quantum system *jumps* into one of the eigenstates of the measured observable.² This jump was thought to be abrupt and take no time at all, it is also known as *reduction of the wave packet*. Some physicists feel quite uneasy about this von Neumann's postulate, to the extent that they reject it either as too primitive (not described by dynamical equations) or as unnecessary. We will come to this point later, in Chapter 3, when we will discuss *piecewise deterministic* stochastic processes that unite both kinds of evolution.

¹ *Emphasized* style will be used in these notes for concepts that are *important*, but will not be *explained*. Sometimes explanation would need too much space, but sometimes because these are either *primitive* or *meta-language* notions.

²Lüders [28] noticed that this formulation is ambiguous in case of degenerate eigenvalues, and generalized it to cover also this situation.

1.2 The Schrödinger Equation

It is rather easy to explain to the mathematician the *Dirac* equation – it became already a part of the mathematical folklore. But Dirac’s equation belongs to *field theory* rather than to *quantum theory*.³ Physicists are being taught in the course of their education rather early that every attempt at a sharp localization of a relativistic particle results in creation and annihilation processes. Sometimes it is phrased as: ”there is no relativistic quantum mechanics – one needs to go directly to Relativistic Quantum Field Theory”. Unfortunately we know of none non-trivial, finite, relativistic quantum field theory in four dimensional space–time continuum. Thus we are left with the mathematics of *perturbation* theory. Some physicists believe that the physical ideas of relativistic quantum field theory are sound, that it is the best theory we ever had, that it is ”the exact description of nature”, that the difficulties we have with it are only temporary, and that they will be overcome one day – the day when bright mathematicians will provide us with new, better, more powerful tools. Some other say: perturbation theory is more than sufficient for all practical purposes, no new tools are needed, that is how *physics is* – so mathematicians better accept it, digest it, and help the physicists to make it more rigorous and to understand what it is really about. Still some other⁴, a minority, also believe that it is only a *temporary* situation, which one day will be resolved. But the resolution will come owing to essentially new physical ideas, and it will result in a new quantum paradigm, more appealing than the present one. It should perhaps be not a surprise if, in an appropriate sense, all these points of view will turn out to be right. In these lectures we will be concerned with the well established Schrödinger equation, which is at the very basis of the current quantum scheme, and with its dissipative generalization – the Liouville equation. In these equations we assume that we know what the *time* is. Such a knowledge is negated in special relativity,⁵ and this results in turn in all kinds of troubles that we are facing since the birth of Einstein’s relativity till this day.⁶

The Schrödinger equation is more difficult than the Dirac one, and this for two reasons: first, it lives on the background of *Galilean* relativity – which have to deal with much more intricate geometric structures than Einstein relativity.⁷ Second, Schrödinger’s equation is about *Quantum Mechanics* and we have to

³In these lectures, ”quantum theory” usually means ”quantum mechanics”, although much of the concepts that we discuss are applicable also to systems with infinite number of degrees of freedom and in particular to quantum field theory.

⁴Including the author of these notes.

⁵There exists however so called ”relativistic Fock–Schwinger proper time formalism” [23, Ch. 2–5–4] where one writes Schrödinger equation with Hamiltonian replaced by ”super–Hamiltonian, and time replaced by ”proper time”

⁶One could try to ”explain” time by saying that there is a preferred time direction selected by the choice of a thermal state of the universe. But that is of no help at all, until we are told how it happens that a particular thermal state is being achieved.

⁷ In group theoretical terms: the proper Lorentz group is simple, while the proper Galilei group is not.

take care about *probabilistic interpretation, observables, states* etc. – which is possible for Schrödinger equation but faces problems in the first-quantized Dirac theory.

Let me first make a general comment about Quantum Theory. There are physicists who would say: quantum theory is about calculating of *Green's Functions* – all numbers of interest can be obtained from these functions and all the other mathematical constructs usually connected with quantum theory are superfluous and unnecessary! It is not my intention to depreciate the achievements of Green's Function Calculators. But for me quantum theory – as any other physical theory – should be about *explaining* things that happen outside of us – as long as such *explanations* are possible. The situation in Quantum Theory today, more than 60 years after its birth, is that Quantum Theory *explains much less* than we would like to have been explained. To reduce quantum theory to Green's function calculations is to reduce its explanatory power almost to zero. It may of course be that in the coming Twenty First Century humanity will unanimously recognize the fact that 'understanding' was a luxury, a luxury of the "primitive age" that is gone for ever. But I think it is worth while to take a chance and to try to understand as much as can be understood in a given situation. Today we are trying to understand Nature in terms of *geometrical pictures* and *random processes*⁸ More specifically, in quantum theory, we are trying to understand in terms of *observables, states, complex Feynman amplitudes* etc. In the next chapter, we will show the way that leads to the Schrödinger Equation using *geometrical language* as much as possible. However, we will not use the machinery of *geometrical quantization* because it treats *time* simply as a *parameter*, and *space* as absolute and given once for all.⁹ On the other hand, geometrical quantization introduces many advanced tools that are unnecessary for our purposes, while at the same time it lacks the concepts which are important and necessary.

Before entering the subject let me tell you the distinguishing feature of the approach that I am advocating, and that will be sketched below in Ch. 2: one obtains a fibration of Hilbert spaces $\mathcal{H} = \bigcup \mathcal{H}_t$ over time. There is a distinguished family of local trivializations, a family parameterized by

- space-time observer
- $U(1)$ gauge.

For each t , the Hilbert space \mathcal{H}_t is a Hilbert space of sections of a complex line bundle over E_t . A space-time observer (that is, a reference frame) allows us to identify the spaces E_t for different t -s, while a $U(1)$ gauge allows us to identify the fibers. Schrödinger's dynamics of a particle in external gravitational and

⁸The paradigm may however change in no so distant future – we may soon try to understand the Universe as a *computing machine*, with geometry replaced by geometry of connections, and randomness replaced by a variant of algorithmic complexity.

⁹Cf. e. g. Ref. [33, Ch. 9].

electromagnetic fields is given by a *Hermitian connection* in \mathcal{H} . Solutions of Schrödinger equation are parallel sections of \mathcal{H} . Thus Schrödinger equation can be written as

$$\nabla\Psi = 0 \tag{1}$$

or, in a local trivialization, as

$$\frac{\partial\Psi(t)}{\partial t} + \frac{i}{\hbar}H(t)\Psi(t) = 0, \tag{2}$$

where $H(t)$ will be a self-adjoint operator in \mathcal{H}_t .¹⁰ Gravitational and electromagnetic forces are coded into this Schrödinger's connection. Let us discuss the resulting structure. First of all there is not a single Hilbert space but a family of Hilbert spaces. These Hilbert spaces can be *identified* either using an *observer* and *gauge* or, better, by using a background dynamical connection. It is only after so doing that one arrives at single-Hilbert-space picture of the textbook Quantum Mechanics – a picture that is the constant source of lot of confusion.¹¹

In Quantum Mechanics we have a *dual* scheme – we use the concepts of *observables* and *states*. We often use the word *measurement* in a mutilated sense of simply *pairing* an observable a with a state ϕ to get the *expected result* – a number $\langle \phi, a \rangle$.

One comment is in place here: to compare the results of actual measurements with predictions of the theory – one needs only *real* numbers. However experience proved that quantum theory with only-real-numbers is inadequate. So, even if the fundamental role of $\sqrt{-1}$ in Quantum Theory is far from being fully understood – we use in Quantum Theory only *complex* Hilbert spaces, complex algebras etc.¹² However, usually, only real numbers are at the end interpreted.

Now, it is not always clear what is understood by *states* and *observables*. There are several possibilities:

As it was already said, it is usual in standard presentations of the quantum theory to *identify* the Hilbert spaces \mathcal{H}_t . There are several options there. Either we identify them according to an observer (+ gauge) or according to the dynamics. If we identify according to the actual dynamics, then states do not change in time – it is always the same state-vector, but observables (like position coordinates) do change in time – we have what is called the Heisenberg picture. If we identify them according to some background "free" dynamics – we have so called *interaction picture*. Or, we can identify Hilbert spaces according to an observer – then observables do not change in time, but state vector is changing

¹⁰A similar idea was mentioned in [3]. For a detailed description of all the constructions – see the forthcoming book [21]

¹¹Notable exceptions can be found in publications from the Genève school of Jauch and Piron.

¹²Quaternionic structures, on the other hand, can be always understood as complex ones with an extra structure – they are unnecessary.

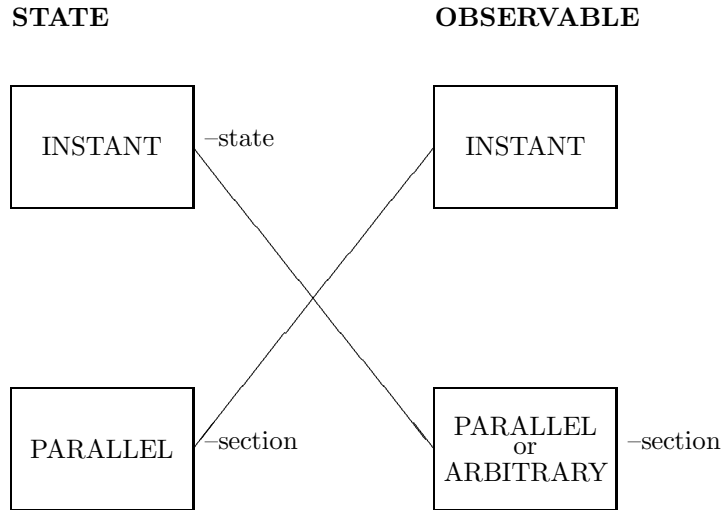


Figure 1: There are several possibilities of understanding of state and observables. They can be *instant*, and thus time-dependent, or they can be *time-sections* – thus time-independent

– we get the Schrödinger picture.

IDENTIFICATION

$$\left\{ \begin{array}{ll} \text{according to the dynamics} & \implies \text{Heisenberg picture} \\ \text{according to an observer} & \implies \text{Schrödinger picture} \end{array} \right.$$

However, there is no reason at all to identify the \mathcal{H}_t -s. Then dynamics is given by parallel transport operators:

$$\begin{aligned}
 U_{t,s} : H_s &\rightarrow H_t, & s &\leq t \\
 U_{t,s} U_{s,r} &= U_{t,r} \\
 U_{t,t} &= id_{\mathcal{H}_t}.
 \end{aligned}$$

1.3 Dissipative Dynamics

The Schrödinger equation describes time evolution of pure states of a quantum system, for instance evolution of pure states of a quantum particle, or of a many body system. Even if these states contain only *statistical information* about most of the physical quantities, the Schrödinger evolution of pure states is *continuous and deterministic*. Under this evolution Hilbert space vector representing the actual state of the system¹³ changes continuously with time, and with it there is a continuous evolution of *probabilities* or *potentialities*, but nothing *happens* – the formalism leaves no place for *events*. Schrödinger equation helps us very little, or nothing at all, to understand *how* potential becomes real. So, if we aim at understanding of this *process of becoming*, if we want to describe it by mathematical equations and to simulate it with computers – we must go beyond Schrödinger’s dynamics. As it happens, we do not have to go very far – it is sufficient to relax only one (but *important*) property of Schrödinger’s dynamics and to admit that pure states can evolve into mixtures. Instead of Schrödinger equation we have then a so called Liouville equation that describes time evolution of mixed states. It contains Schrödinger equation as a special case.¹⁴ It was shown in [11] that using the Liouville type of dynamics it is possible to describe coupling between quantum systems and classical degrees of freedom of measurement devices. One can derive also a piecewise deterministic random process that takes place on the manifold of pure states. In this way one obtains a minimal description of ”quantum jumps” (or ”reduction of wave packets”) and accompanying, directly observable jumps of the coupled classical devices. In Ch. 3 simple models of such couplings will be discussed. The interested reader will find more examples in Refs. [8]–[11].¹⁵ In particular in [11] the most advanced model of this kind, the SQUID–tank model is discussed in details.

2 Geometry of Schrödinger’s Equation

¹³Some physicists deny ”objectivity” of quantum states – they would say that Hilbert space vectors describe not states of the system, but states of *knowledge* or *information* about the system. In a recent series of papers (see [1] and references therein) Aharonov and Vaidman [1] attempt to justify objectivity of quantum states. Unfortunately their arguments contain a loophole.

¹⁴ It should be noted, however, that Schrödinger equation describes evolution of state *vectors*. and thus contains direct information about *phases*. This information is absent in the Liouville equation, and its restoration (e. g. as it is with the *Berry phase*) may sometimes create a non–trivial task.

¹⁵Cf. also the recent (June 1994) paper ”Particle Tracks, Events and Quantum Theory”, by the author.

2.1 Preliminaries

Galilean General Relativity is a theory of space–time structure, gravitation and electromagnetism based on the assumption of existence of an absolute time function. Many of the cosmological models based on Einstein’s relativity admit also a distinguished time function. Therefore Galilean physics is not evidently wrong. Its predictions must be tested by experiments. Galilean relativity is not that elegant as the one of Einstein. This can be already seen from the group structures: the homogeneous Lorentz group is *simple*, while the homogeneous Galilei group is a semidirect product of the rotation group and of three commuting boosts. Einstein’s theory of gravitation is based on one metric tensor, while Galilean gravity needs both: space metric and space–time connection. Similarly for quantum mechanics: it is rather straightforward to construct generally covariant wave equations for Einstein’s relativity, while general covariance and geometrical meaning of the Schrödinger equation was causing problems, and it was not discussed in textbooks. In the following sections we will present a brief overview of some of these problems.

2.2 Galilean General Relativity

Let us discuss briefly geometrical data that are needed for building up generally covariant Schrödinger’s equation. More details can be found in Ref. [21].¹⁶

Our space–time will be a refined version of that of Galilei and of Newton, i.e. space–time with absolute simultaneuity. Four dimensional space–time E is fibred over one–dimensional time B . The fibers E_t of E are three–dimensional Riemannian manifolds, while the basis B is an affine space over \mathbf{R} . By a coordinate system on E we will always mean a coordinate system $x^\mu = (x^0, x^i)$, $i = 1, 2, 3$, *adapted to the fibration*. That means: any two events with the same coordinate x^0 are simultaneous, i.e. in the same fibre of E .

Coordinate transformations between any two adapted coordinate systems are of the form:

$$\begin{aligned}x^{0'} &= x^0 + \text{const}, \\x^{i'} &= x^{i'}(x^0, x^i).\end{aligned}$$

We will denote by β the time form dx^0 . Thus in adapted coordinates $\beta_0 = 1, \beta_i = 0$.

E is equipped with a contravariant *degenerate* metric tensor which, in adapted

¹⁶The reader may also consult [19], where a different approach, using dimensional reduction along a null Killing vector, is discussed.

coordinates, takes the form

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & g^{ij} \end{array} \right).$$

where g^{ij} is of signature $(+++)$. We denote by g_{ij} the inverse 3×3 matrix. It defines Riemannian metric in the fibers of E .

We assume a torsion-free connection in E that preserves the two geometrical objects $g^{\mu\nu}$ and β .¹⁷ The condition $\nabla\beta = 0$ is equivalent to the conditions $\Gamma_{\mu\nu}^0 = 0$ on the connection coefficients. Let us introduce the notation $\Gamma_{\mu\nu,i} = g_{ij}\Gamma_{\mu\nu}^j$. Then $\nabla g^{\mu\nu} = 0$ is equivalent to the equations:

$$\partial_\mu g_{ij} = \Gamma_{\mu i,j} + \Gamma_{\mu j,i}. \quad (3)$$

Then, because of the assumed zero torsion, the space part of the connection can be expressed in terms of the space metric in the Levi-Civita form:

$$\Gamma_{ij,k} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \quad (4)$$

From the remaining equations:

$$\partial_0 g_{ij} = \Gamma_{0i,j} + \Gamma_{0j,i} \quad (5)$$

we find that the (i,j) -symmetric part of $\Gamma_{0i,j}$ is equal to $\frac{1}{2}\partial_0 g_{ij}$, otherwise the connection is undetermined. We can write it, introducing a new geometrical object Φ , as

$$\Gamma_{i0,j} = \frac{1}{2} (\partial_0 g_{ij} + \Phi_{ij}), \quad (6)$$

$$\Gamma_{00,j} = \Phi_{0j}, \quad (7)$$

where $\Phi_{\mu\nu} = -\Phi_{\nu\mu}$ is antisymmetric. Notice that Φ is not a tensor, except for pure space transformations or time translations.

2.2.1 The Bundle of Galilei Frames

A basis e_μ in TE is called a Galilei frame if $e_0 = \partial_0$, and if e_i are unit space-like vectors. If e_μ and \tilde{e}_μ are two Galilei frames at the same space-time point, then they are related by a transformation of the homogeneous Galilei group G :

$$\tilde{e}_0 = e_0 + \mathbf{e} \cdot \mathbf{v}, \quad (8)$$

¹⁷Some of these assumptions are superfluous as they would follow anyhow from the assumption $d\Omega = 0$ in the next paragraph.

$$\tilde{\mathbf{e}} = \mathbf{e}\mathbf{\Lambda}, \quad (9)$$

where $\mathbf{v} \in \mathbf{R}^3$ and $\mathbf{\Lambda}$ is an orthogonal 3×3 matrix. The bundle of Galilei frames is a principal G bundle.

2.2.2 The Bundle of Velocities

The homogeneous Galilei group G acts on \mathbf{R}^3 in two natural ways: by linear and by affine transformations. The first action is not *effective* one – it involves only the rotations:

$$(\mathbf{\Lambda}, \mathbf{v}) : \mathbf{x} \mapsto \mathbf{\Lambda}\mathbf{x}. \quad (10)$$

The bundle associated to this action can be identified with the vertical subbundle of TE – i.e. with the bundle VE of vectors tangent to the fibers of $E \rightarrow B$. G acts also on \mathbf{R}^3 by affine isometries :

$$(\mathbf{\Lambda}, \mathbf{v}) : \mathbf{y} \mapsto \mathbf{\Lambda}\mathbf{y} + \mathbf{v}. \quad (11)$$

To this action there corresponds an associated bundle, which is an affine bundle over the vector bundle VE . It can be identified with the subbundle of TE consisting of vectors ξ tangent to E , and such that $\beta(\xi) = 1$ or, equivalently, as the bundle of *first jets of sections* of $E \rightarrow B$. We will call it J_1E .

We will denote by (x^0, x^i, y_0^i) the coordinates in J_1E corresponding to coordinates x^μ of E .

2.3 The Presymplectic Form

The connection Γ can be also considered as a principal connection in the bundle of Galilei frames. It induces an affine connection in the affine bundle $J_1E \xrightarrow{\pi} E$. As a result, it defines a natural VE -valued one-form ν_Γ on J_1E . It can be described as follows: given a vector ξ tangent to J_1E at (x, y_0) it projects onto $d\pi(\xi)$. Then $\nu_\Gamma(\xi)$ is defined as the difference of ξ and the horizontal lift of $d\pi(\xi)$. It is a vertical tangent vector to J_1E , and can be identified with an element of VE . In coordinates:

$$\nu_\Gamma^i = dy_0^i + \left(\Gamma_{\mu j}^i y_0^j + \Gamma_{\mu 0}^i \right) dx^\mu. \quad (12)$$

There is another VE -valued one-form on J_1E , namely the canonical form θ . Given ξ at (x, y_0) , we can decompose ξ into space- and time-component along y_0 . Then $\theta(\xi)$ is defined as its space component. In coordinates:

$$\theta^i = dx^i - y_0^i dx^0. \quad (13)$$

Then, because the fibers of VE are endowed with metric g^{mn} , we can build out the following important two-form Ω on J_1E :

$$\Omega = g_{mn} \nu_\Gamma^m \wedge \theta^n. \quad (14)$$

Explicitly

$$\begin{aligned} \Omega = g_{lm} \left[dy_0^l \wedge \theta^m + \left(\Gamma_{jk}^l y_0^k + \Gamma_{j0}^l \right) \theta^j \wedge \theta^m + \right. \\ \left. + \left(\Gamma_{jk}^l y_0^j y_0^k + \Gamma_{0k}^l y_0^k + \Gamma_{k0}^l y_0^k + \Gamma_{00}^l \right) dx^0 \wedge \theta^m \right] \end{aligned} \quad (15)$$

The following theorem, proven in [21], gives a necessary and sufficient condition for Ω to be closed.

Theorem 1 *The following conditions (i–iii) are equivalent:*

- (i) $d\Omega = 0$,
- (ii) $R_{\nu\rho}^{\mu\sigma} = R_{\rho\nu}^{\sigma\mu}$ where $R_{\mu\nu}^{\sigma\rho}$ is the curvature tensor of Γ and $R_{\nu\rho}^{\mu\sigma} = g^{\mu\lambda} R_{\lambda\nu}^{\sigma\rho}$
- (iii) $\partial_{[\mu} \Phi_{\nu\sigma]} = 0$.¹⁸

2.4 Quantum Bundle and Quantum Connection

Let Q be a principal $U(1)$ bundle over E and let Q^\uparrow be its pullback to $J_1 E$. We denote by P and P^\uparrow the associated Hermitian line bundles corresponding to the natural action of $U(1)$ on \mathbb{C} . There is a special class of principal connections on Q^\uparrow , namely those whose connection forms vanish on vectors tangent to the fibers of $Q^\uparrow \rightarrow Q$. As has been discussed in [29] specifying such a connection on Q^\uparrow is equivalent to specifying a *system of connections* on Q parameterized by the points in the fibers of $Q^\uparrow \rightarrow Q$. Following the terminology of [29] we call such a connection *universal*.¹⁹ The fundamental assumption that leads to the Schrödinger equations reads as follows:

Quantization Assumption: *There exists a universal connection ω on Q^\uparrow whose curvature is $i\Omega$.*²⁰

We call such an ω a *quantum connection*. From the explicit form of Ω one can easily deduce that ω is necessarily of the form

$$\omega = i(d\phi + a_\mu dx^\mu)$$

where $0 \leq \phi \leq 2\pi$ parameterizes the fibres of Q ,

$$a_0 = -\frac{1}{2} y_0^2 + \alpha_0,$$

$$a_i = g_{ij} y_0^j + \alpha_i,$$

and $\alpha^\nu = (\alpha^0, \alpha^i)$ is a local potential for Φ .

¹⁸Notice that because Φ is not a tensor, the last condition need not be, a priori, generally covariant.

¹⁹i.e. *universal* for the system of connections.

²⁰We choose the physical units in such a way that the Planck constant \hbar and mass of the quantum particle m are equal to 1.

2.5 Schrödinger's Equation and Schrödinger's Bundle

As it is shown in [21], there exists a natural $U(1)$ -invariant metric on P of signature $(++++-)$. Explicitly

$$\begin{aligned} \overset{5}{g} &= d\phi \otimes dx^0 + dx^0 \otimes d\phi \\ &+ g_{ij} dx^i \otimes dx^j + 2\alpha_0 dx^0 \otimes dx^0 + \\ &+ \alpha_i (dx^i \otimes dx^0 + dx^0 \otimes dx^i) . \end{aligned}$$

Using this metric we can build out a natural Lagrangian for equivariant functions $\psi : P \rightarrow \mathbb{C}$ or, equivalently, for sections of the line bundle Q . The Euler–Cartan equation for this Lagrangian will prove to be nothing but the Schrödinger equation. Notice that the action of $U(1)$ group on P defines an Killing vector field for $\overset{5}{g}$ which is isotropic. Therefore the above construction can explain why the approach of [19] works.

More precisely, the construction leading to the generally covariant Schrödinger–Pauli equation for a charged spin 1/2 particle in external gravitational and electromagnetic field can be described as follows.

The contravariant metric $\overset{5}{g}^{-1} = (g^{\alpha\beta})$, $\alpha, \beta = 0, 1, 2, 3, 5$,

$$(g^{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g^{ij} & -g^{ij}a_j \\ 1 & -g^{ij}a_j & \mathfrak{a}^2 - 2a_0 \end{pmatrix} . \quad (16)$$

can be obtained from the following Clifford algebra of 4×4 complex matrices:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ \gamma^5 &= \begin{pmatrix} -\underline{\sigma} \cdot \underline{a} & 1 \\ -2a_0 & \underline{\sigma} \cdot \underline{a} \end{pmatrix} \end{aligned} \quad (17)$$

One takes then 5-dimensional charged Dirac operator $\gamma^\alpha \nabla_\alpha$ and considers spinors that are equivariant with respect to the fifth coordinate $x^5 = \phi$:

$$\frac{\partial \psi}{\partial \phi} = -i\psi . \quad (18)$$

This first-order, four-component spinor (called Lévy–Leblond equation in Ref. [24]) equation reduces then easily to the second-order, two-component Schrödinger Pauli equation with the correct Landé factor.²¹

We finish this section with pointing to the Ref. [21], where Schrödinger’s quantization is discussed in details and where a probabilistic interpretation of generally covariant Schrödinger equation is given using the bundle $L^2(E_t)$ of Hilbert spaces. The parallel transport induced by the quantum connection is shown in [21] to be directly related to Feynman amplitudes.

3 Coupled Quantum and Classical Systems

3.1 Preliminaries

Replacing Schrödinger’s evolution, which governs the dynamics of pure states, by an equation of the Liouville type, that describes time evolution of mixed states, is a necessary step – but it does not suffice for modeling of *real world events*. One must take, to this end, two further steps. First of all we should admit that in our reasoning, our communication, our description of *facts* – we are using classical logic. Thus *somewhere* in the final step of transmission of information from quantum systems to macroscopic recording devices and further, to our senses and minds, a *translation* between quantum and classical should take place. That such a translation is necessary is evident also when we consider the opposite direction: to *test* a physical theory we perform *controlled experiments*. But some of the *controls* are always of classical nature – they are external parameters with concrete numerical values. So, we need to consider systems with both quantum and classical degrees of freedom, and we need evolution equations that enable communication in both directions, i.e. :

- *flow of information from quantum to classical*
and
- *control of quantum states and processes by classical parameters* .

3.2 Completely Positive Maps

We begin with a brief recall of relevant mathematical concepts. Let \mathcal{A} be a C^* –algebra. We shall always assume that \mathcal{A} has unit I . An element $A \in \mathcal{A}$ is *positive*, $A \geq 0$, iff it is of the form B^*B for some $B \in \mathcal{A}$. Every element of a C^* –algebra is a linear combination of positive elements. A linear functional $\phi : \mathcal{A} \rightarrow \mathbb{C}$ is *positive* iff $A \geq 0$ implies $\phi(A) \geq 0$. Every positive functional on a C^* –algebra is continuous and $\|\phi\| = \phi(I)$. Positive functionals of norm one are called *states*. The space of states is a convex set. Its extremal points are called *pure states*. The canonical *GNS construction* allows one to associate

²¹For an alternative detailed derivation see [12]

with each state ω a representation π_ω of \mathcal{A} on a Hilbert space \mathcal{H}_ω , and a cyclic vector $\Psi_\omega \in \mathcal{H}_\omega$ such that $(\Psi_\omega, \pi_\omega(A)\Psi_\omega) = \omega(A)$, $A \in \mathcal{A}$. Irreducibility of π_ω is then equivalent to purity of ω .

Quantum theory gives us a powerful formal language and statistical algorithms for describing general physical systems. Physical *quantities* are coded there by Hermitian elements of a C^* -algebra \mathcal{A} of *observables*, while information about their values (quantum algorithms deal, in general, only with statistical information) is coded in states of \mathcal{A} . Pure states correspond to a maximal possible information. For each state ω , and for each $A = A^* \in \mathcal{A}$ the (real) number $\omega(A)$ is interpreted as *expectation value* of observable A in state ω , while

$$\delta_\omega^2(A) \doteq \omega((A - \omega(A))^2) = \omega(A^2) - (\omega(A))^2$$

is the *quadratic dispersion* of A in the state ω . It is assumed that repeated *measurements* of A ²² made on systems *prepared in a state* ω will give a sequence of values a_1, \dots, a_n so that *approximately* $\frac{1}{N} \sum_{i=1}^N a_i \approx \omega(A)$, and $\frac{1}{N} \sum (a_i)^2 - (\frac{1}{N} \sum a_i)^2 \approx \delta_\omega^2$. If \mathcal{A} is Abelian, then it is isomorphic to an algebra of functions $\mathcal{A} \approx C(X)$. Then pure states of \mathcal{A} are dispersion free – they are parameterized by points $x \in X$ and we have $\omega_x(A) = A(x)$. This corresponds to a *classical theory*: all observables mutually commute and maximal possible information is without any statistical dispersion. In the extreme opposition to that is *pure quantum theory* – here defined as that one when \mathcal{A} is a factor, that is has a trivial centre. The centre $\mathcal{Z}(\mathcal{A})$ of a C^* -algebra \mathcal{A} is defined as $\mathcal{Z}(\mathcal{A}) = \{C \in \mathcal{A} : AC = CA, A \in \mathcal{A}\}$. In general $\mathbb{C} \cdot I \subset \mathcal{Z}(\mathcal{A}) \subset \mathcal{A}$. If $\mathcal{Z}(\mathcal{A}) = \mathbb{C} \cdot I$ – we have pure classical theory. If $\mathcal{Z}(\mathcal{A}) = \mathbb{C} \cdot I$ – we have pure quantum theory. In between we have a theory with *superselection rules*. Many physicists believe that the "good theory" should be a "pure quantum" theory. But I know of *no one good reason* why this should be the case. In fact, we will see that cases with a nontrivial $\mathcal{Z}(\mathcal{A})$ are interesting ones. Of course, one can always argue that whenever we have an algebra with a nontrivial centre – it is a subalgebra of an algebra with a trivial one, for instance of $B(\mathcal{H})$ – the algebra of all bounded operators on some Hilbert space. This is, however, not a good argument – one could argue as well that we do not need to consider different groups as most of them are subgroups of $\mathcal{U}(\mathcal{H})$ – the unitary group of an infinite dimensional Hilbert space – so why to bother with others?

Let \mathcal{A}, \mathcal{B} be C^* -algebras. A linear map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is *Hermitian* if $\phi(A^*) = \phi(A)^*$. It is *positive* iff $A \geq 0$, $A \in \mathcal{A}$ implies $\phi(A) \geq 0$. Because Hermitian elements of a C^* -algebra are differences of two positive ones – each positive map is automatically Hermitian. Let \mathcal{M}_n denote the n by n matrix algebra, and let $\mathcal{M}_n(\mathcal{A}) = \mathcal{M}_n \otimes \mathcal{A}$ be the algebra of $n \times n$ matrices with entries from

²²Our point is that "measurement" is an undefined concept in standard quantum theory, and that the probabilistic interpretation must be, because of that, brought from outside. What we propose is to *define* measurement as a CP semigroup coupling between a classical and a quantum system and to *derive* the probabilistic interpretation of the quantum theory from that of the classical one.

\mathcal{A} . Then $\mathcal{M}_n(\mathcal{A})$ carries a natural structure of a C^* -algebra. With respect to this structure a matrix $\mathbf{A} = (A_{ij})$ from $\mathcal{M}_n(\mathcal{A})$ is positive iff it is a sum of matrices of the form $(A_{ij}) = (A_i^* A_j)$, $A_i \in \mathcal{A}$. If \mathcal{A} is an algebra of operators on a Hilbert space \mathcal{H} , then $\mathcal{M}_n(\mathcal{A})$ can be considered as acting on $\mathcal{H}^n \doteq \mathcal{H} \otimes C^n = \oplus_{i=1}^n \mathcal{H}$. Positivity of $\mathbf{A} = (A_{ij})$ is then equivalent to $(\Psi, \mathbf{A}\Psi) \geq 0$, $\Psi \in \mathcal{H}^n$, or equivalently, to $\sum_{i,j} (\Psi_i, A_{ij}\Psi_j) \geq 0$ for all $\Psi_1, \dots, \Psi_n \in \mathcal{H}$.

A positive map ϕ is said to be *completely positive* or, briefly, *CP* iff $\phi \otimes id_n : \mathcal{A} \otimes \mathcal{M}_n \rightarrow \mathcal{B} \otimes \mathcal{M}_n$ defined by $(\phi \otimes id_n)(A \otimes M) = \phi(A) \otimes M$, $M \in \mathcal{M}_n$, is positive for all $n = 2, 3, \dots$. When written explicitly, complete positivity is equivalent to

$$\sum_{i,j=1}^n B_i^* \phi(A_i^* A_j) B_j \geq 0 \quad (19)$$

for every $A_1, \dots, A_n \in \mathcal{A}$ and $B_1, \dots, B_n \in \mathcal{B}$. In particular every homomorphism of C^* algebras is completely positive. One can also show that if either \mathcal{A} or \mathcal{B} is Abelian, then positivity implies complete positivity. Another important example: if \mathcal{A} is a C^* algebra of operators on a Hilbert space \mathcal{H} , and if $V \in \mathcal{B}(\mathcal{H})$, then $\phi(A) = VAV^*$ is a CP map $\phi : \mathcal{A} \rightarrow \phi(\mathcal{A})$. The celebrated Stinespring theorem gives us a general form of a CP map. Stinespring's construction can be described as follows. Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a CP map. Let us restrict to a case when $\phi(I) = I$. Let \mathcal{B} be realized as a norm closed algebra of bounded operators on a Hilbert space \mathcal{H} . One takes then the algebraic tensor product $\mathcal{A} \otimes \mathcal{H}$ and defines on this space a sesquilinear form \langle, \rangle by

$$\langle A \otimes \Psi, A' \otimes \Psi' \rangle = (\Psi, \phi(A^* A') \Psi'). \quad (20)$$

This scalar product is then positive semi-definite because of complete positivity of ϕ . Indeed, we have

$$\langle \sum_i A_i \otimes \Psi_i, \sum_j A_j \otimes \Psi_j \rangle = \sum_{i,j} (\Psi_i, \phi(A_i^* A_j) \Psi_j) \geq 0. \quad (21)$$

Let \mathcal{N} denote the kernel of \langle, \rangle . Then $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$ is a pre-Hilbert space. One defines a representation π of \mathcal{A} on $\mathcal{A} \otimes \mathcal{H}$ by $\pi(A) : A' \otimes \Psi \mapsto AA' \otimes \Psi$. One shows then that \mathcal{N} is invariant under $\pi(\mathcal{A})$, so that π goes to the quotient space. Similarly, the map $\mathcal{H} \ni \Psi \mapsto I \otimes \Psi \in \mathcal{A} \otimes \mathcal{H}$ defines an isometry $V : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H} / \mathcal{N}$. We get then $\phi(A) = V^* \pi(A) V$ on the completion \mathcal{H}_ϕ of $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$.

Theorem 2 (Stinespring's Theorem) *Let \mathcal{A} be a C^* -algebra with unit and let $\phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be a CP map. Then there exists a Hilbert space \mathcal{H}_ϕ , a representation π_ϕ of \mathcal{A} on \mathcal{H}_ϕ , and a bounded linear map $V : \mathcal{H} \rightarrow \mathcal{H}_\phi$ such that*

$$\phi(A) = V^* \pi_\phi(A) V. \quad (22)$$

V is an isometry iff ϕ is unital i.e. iff ϕ maps the unit of \mathcal{A} into the identity operator of \mathcal{H} . If \mathcal{A} and \mathcal{H} are separable, then \mathcal{H}_ϕ can be taken separable.

The space of CP maps from \mathcal{A} to $\mathcal{B}(\mathcal{H})$ is a convex set. Arveson [2] proved that ϕ is an extremal element of this set iff the representation π_ϕ above is irreducible.

3.3 Dynamical Semigroups

A *dynamical semigroup* on a C^* -algebra of operators \mathcal{A} is a strongly continuous semigroup of CP maps of \mathcal{A} into itself. A semigroup α_t , is norm continuous iff its infinitesimal generator L is bounded as a linear map $L : \mathcal{A} \rightarrow \mathcal{A}$. We then have

$$\alpha_t = \exp(tL), t \geq 0. \quad (23)$$

The right hand side is, in this case, a norm convergent series *for all* real values of t , however for t negative the maps $\exp(tL) : \mathcal{A} \rightarrow \mathcal{A}$, although Hermitian, need not be positive.

Evolution of observables gives rise, by duality, to evolution of positive functionals. One defines $\alpha^t(\phi)(A) = \phi(\alpha_t(A))$. Then α_t preserves the unit of \mathcal{A} iff α^t preserves normalization of states. A general form of a generator of a dynamical semigroup in finite dimensional Hilbert space has been derived by Gorini, Kossakowski and Sudarshan [22], and Lindblad [27] gave a general form of a bounded generator of a dynamical semigroup acting on the algebra of all bounded operators $B(\mathcal{H})$. It is worthwhile to cite, after Lindblad, his original motivation:

”The dynamics of a finite closed quantum system is conventionally represented by a one-parameter group of unitary transformations in Hilbert space. This formalism makes it difficult to describe irreversible processes like the decay of unstable particles, approach to thermodynamic equilibrium and measurement processes [...]. It seems that the only possibility of introducing an irreversible behaviour in a finite system is to avoid the unitary time development altogether by considering non-Hamiltonian systems. ”

In a recent series of papers [7, 8, 9, 10] Ph. Blanchard and the present author were forced to introduce dynamical semigroups because of another difficulty, namely because of impossibility of obtaining a nontrivial Hamiltonian coupling of classical and quantum degrees of freedom in a system described by an algebra with a non-trivial centre. We felt that lack of a dynamical understanding of quantum mechanical probabilistic postulates is more than annoying. We also believed that the word ”measurement” instead of being banned, as suggested by J. Bell [5, 6], can be perhaps given a precise and acceptable meaning. We suggested that *a measurement process is a coupling of a quantum and of a classical system, where information about quantum state is transmitted to*

the classical recording device by a dynamical semigroup of the total system . It is instructive to see that such a transfer of information can not indeed be accomplished by a Hamiltonian or, more generally, by any automorphic evolution²³. To this end consider an algebra \mathcal{A} with centre \mathcal{Z} . Then \mathcal{Z} describes classical degrees freedom. Let ω be a state of \mathcal{A} , then $\omega|_{\mathcal{Z}}$ denotes its restriction to \mathcal{Z} . Let α_t be an automorphic evolution of \mathcal{A} , and denote $\omega_t = \alpha^t(\omega)$. Each α_t is an automorphism of the algebra \mathcal{A} , and so it leaves its centre invariant: $\alpha_t : \mathcal{Z} \rightarrow \mathcal{Z}$. The crucial observation is that, because of that fact, the restriction $\omega_t|_{\mathcal{Z}}$ depends only on $\omega_0|_{\mathcal{Z}}$, as the evolution of states of \mathcal{Z} is dual to the evolution of the observables in \mathcal{Z} . This shows that information transfer from the total algebra \mathcal{A} to its centre \mathcal{Z} is impossible – unless we use more general, non-automorphic evolutions.

From the above reasoning it may be seen that the Schrödinger picture, when time evolution is applied to states, is better adapted to a discussion of information transfer between different systems. The main properties that a dynamical semigroup α^t describing time evolution of states should have are: α^t should preserve convex combinations, positivity and normalization. One can demand even more – it is reasonable to demand a special kind of stability: that it should be always possible to extend the system and its evolution in a trivial way, by adding extra degrees of freedom that do not couple to our system.²⁴ That is exactly what is assured by *complete positivity* of the maps α_t . One could also think that we should require even more, namely that α^t transforms pure states into pure states. But to assume that would be already too much, as one can prove that then α^t must be dual to an automorphic evolution. It appears that information gain in one respect (i.e. learning about the actual state of the quantum system) must be accompanied by information loss in another one – as going from pure states to mixtures implies entropy growth.

We will apply the theory of dynamical semigroup to algebras with a non-trivial centre. In all our examples we will deal with tensor products of $\mathcal{B}(\mathcal{H})$ and an Abelian algebra of functions. The following theorem by Christensen and Evans [14] generalizes the results of Gorini, Kossakowski and Sudarshan and of Lindblad to the case of arbitrary C^* -algebra.

Theorem 3 (Christensen – Evans) *Let $\alpha_t = \exp(Lt)$ be a norm-continuous semigroup of CP maps of a C^* - algebra of operators $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. Then there exists a CP map ϕ of \mathcal{A} into the ultraweak closure $\bar{\mathcal{A}}$ and an operator $K \in \bar{\mathcal{A}}$ such that the generator L is of the form:*

$$L(A) = \phi(A) + K^*A + AK. \tag{24}$$

²³For a discussion of this fact in a broader context of algebraic theory of superselection sectors – cf. Landsman [25, Sec. 4. 4]. Cf. also the no-go result by Ozawa [31]

²⁴That requirement is also necessary to guarantee physical consistency of the whole framework, as we always neglect some degrees of freedom as either irrelevant or yet unknown to us.

We will apply this theorem to the cases of \mathcal{A} being a von Neumann algebra, and the maps α_t being *normal*. Then ϕ can be also taken normal. We also have $\bar{\mathcal{A}} = \mathcal{A}$, so that $K \in \mathcal{A}$. We will always assume that $\alpha_t(I) = I$ or, equivalently, that $L(I) = 0$. Moreover, it is convenient to introduce $H = i(K - K^*)/2 \in \mathcal{A}$, then from $L(I) = 0$ we get $K + K^* = -\phi(I)$, and so $K = -iH - \phi(1)/2$. Therefore we have

$$L(A) = i[H, A] + \phi(A) - \{\phi(1), A\}/2, \quad (25)$$

where $\{, \}$ denotes anticommutator. Of particular interest to us will be generators L for which ϕ is extremal²⁵. By the already mentioned result of Arveson [2] this is the case when ϕ is of the form

$$\phi(A) = V^* \pi(A) V, \quad (26)$$

where π is an irreducible representation of \mathcal{A} on a Hilbert space \mathcal{K} , and $V : \mathcal{H} \rightarrow \mathcal{K}$ is a bounded operator (it must be, however, such that $V^* \mathcal{A} V \subset \mathcal{A}$).

3.4 Coupling of Classical and Quantum Systems

We consider a model describing a coupling between a quantum and a classical system. To concentrate on main ideas rather than on technical details let us assume that the quantum system is described in an n -dimensional Hilbert space \mathcal{H}_q , and that it has as its algebra of observables $\mathcal{B}(\mathcal{H}_q) \approx \mathcal{M}_n$. Similarly, let us assume that the classical system has only a finite number of pure states $\mathcal{S} = \{s_1, \dots, s_m\}$. Its algebra of observables \mathcal{A}_{cl} is then isomorphic to \mathbb{C}^m . For the algebra of the total system we take $\mathcal{A}_{tot} = \mathcal{A}_q \otimes \mathcal{A}_{cl}$ which is isomorphic to the diagonal subalgebra of $\mathcal{M}_m(\mathcal{A}_q)$. Observables of the total system are block diagonal matrices:

$$\mathbf{A} = \text{diag}(A_\alpha) = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & A_m \end{pmatrix},$$

where A_α , $(\alpha = 1, \dots, m)$ are operators in \mathcal{H}_q .²⁶ Both \mathcal{A}_q and \mathcal{A}_{cl} can be considered as subalgebras of \mathcal{A}_{tot} consisting respectively of matrices of the form $\text{diag}(A, \dots, A)$, $A \in \mathcal{A}_q$ and $\text{diag}(\lambda_1 I_n, \dots, \lambda_m I_n)$, $\lambda_\alpha \in \mathbb{C}$. States of the quantum system are represented by positive, trace one, operators on $\mathcal{B}(\mathcal{H}_q)$. States of the classical system are m -tuples of non-negative numbers p_1, \dots, p_m , with $\sum_\alpha p_\alpha = 1$. States of the total system are represented by block diagonal matrices $\rho = \text{diag}(\rho_1, \dots, \rho_m)$, with $\mathcal{B}(\mathcal{H}_q) \ni \rho_\alpha \geq 0$ and $\sum_\alpha \text{Tr}(\rho_\alpha) = 1$. For the

²⁵It should be noticed, however, that splitting of L into ϕ and K is, in general, not unique – cf. e. g. Refs [15] and [32, Ch. III. 29–30].

²⁶It is useful to have the algebra \mathcal{A}_{tot} represented in such a form, as it enables us to apply the theorem of Christensen–Evans.

expectation value we have $\rho(\mathbf{A}) = \sum_{\alpha} Tr(\rho_{\alpha} A_{\alpha})$. Given a state ρ of the total system, we can trace over the quantum system to get an effective state of the classical system $p_{\alpha} = Tr(\rho_{\alpha})$, or we can trace over the classical system to get the effective state of the quantum system $\hat{\rho} = \sum_{\alpha} \rho_{\alpha}$.²⁷

Let us consider dynamics. Since the classical system has a discrete set of pure states, there is no non-trivial and continuous time evolution for the classical system that would map pure states into pure states. As for the quantum system, we can have a Hamiltonian dynamics, with the Hamiltonian possibly dependent on time and on the state of the classical system $H(t) = diag(H(\alpha, t))$. As we already know a non-trivial coupling between both systems is impossible without a dissipative term, and the simplest dissipative coupling is of the form $L(\mathbf{A}) = V\pi(\mathbf{A})V^*$, where π is an irreducible representation of the algebra \mathcal{A}_{tot} in a Hilbert space \mathcal{H}_{π} , and $V : \mathcal{H}_q \rightarrow \mathcal{H}_{\pi}$ is a linear map. It is easy to see that such an $L(A)$ is necessarily of the form:

$$L(\mathbf{A}) = \mathbf{V}^* \mathbf{A} \mathbf{V},$$

where \mathbf{V} is an $m \times m$ block matrix with only one non-zero entry. A more general CP map of \mathcal{A}_{tot} is of the same form, but with \mathbf{V} having at most one non-zero element in each of its rows.

Let us now discuss desired couplings in somewhat vague, but more intuitive, physical terms. We would like to write down a coupling that enables transfer of information from quantum to classical system. There may be many ways of achieving this aim – the subject is new and there is no ready theory that fits. We will see however that a naive description of a coupling works quite well in many cases. The idea is that the simplest coupling *associates* to a *property* of the quantum system a *transformation* of the actual state of the classical system. *Properties* are, in quantum theory, represented by projection operators. Sometimes one considers also more general, *unsharp* or *fuzzy* properties. They are represented by positive elements of the algebra which are bounded by the unit. A *measurement* should discriminate between mutually exclusive and exhaustive properties. Thus one usually considers a family of mutually orthogonal projections e_i of sum one. With an *unsharp measurement* one associates a family of positive elements a_i of sum one.

As there is no yet a complete, general, theory of dissipative couplings of classical and quantum systems, the best we can do is to show some characteristic examples. It will be done in the following section. For every example a piecewise deterministic random process will be described that takes place on the space of pure states of the total system²⁸ and which reproduces the Liouville evolu-

²⁷One can easily imagine a more general situation when tracing over the classical system will not be meaningful. This can happen if we deal with several *phases* of the quantum system, parameterized by the classical parameter α . It may then happen that the total algebra is not the tensor product algebra. For instance, instead of one Hilbert space \mathcal{H}_q , we may have, for each value of α , a Hilbert space $\mathcal{H}_{q,\alpha}$ of dimension n_{α} .

²⁸One may wonder what does that mean mathematically, as the space of pure states of a C^* algebra is, from measure-theoretical point of view, a rather unpleasant object. The answer is

tion of the total system by averaging over the process. A theory of piecewise deterministic (PD) processes is described in a recent book by M. H. Davis [15]. Processes of that type, but without a non-trivial evolution of the classical system, were discussed also in physical literature – cf. Refs [13, 17, 18, 20].²⁹ We will consider Liouville equations of the form

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_i \left(V_i \rho(t) V_i^* - \frac{1}{2} \{V_i^* V_i, \rho(t)\} \right), \quad (27)$$

where in general H and the V_i can explicitly depend on time. The V_i will be chosen as tensor products $V_i = \sqrt{\kappa} e_i \otimes \phi_i$, where ϕ_i act as transformations³⁰ on classical (pure) states.

3.5 Examples of Classical–Quantum Couplings

3.5.1 The Simplest Coupling

First, we consider only one orthogonal projector e on the two-dimensional Hilbert space $\mathcal{H}_q = \mathbb{C}^2$. To define the dynamics we choose the coupling operator V in the following way:

$$V = \sqrt{\kappa} \begin{pmatrix} 0, & e \\ e, & 0 \end{pmatrix}. \quad (28)$$

The Liouville equation (27) for the density matrix $\rho = \text{diag}(\rho_1, \rho_2)$ of the total system reads now

$$\begin{aligned} \dot{\rho}_1 &= -i[H, \rho_1] + \kappa(e\rho_2e - \frac{1}{2}\{e, \rho_1\}), \\ \dot{\rho}_2 &= -i[H, \rho_2] + \kappa(e\rho_1e - \frac{1}{2}\{e, \rho_2\}). \end{aligned} \quad (29)$$

For this particularly simple coupling the effective quantum state $\hat{\rho} = \pi_q(\rho) = \rho_1 + \rho_2$ evolves independently of the state of the classical system. One can say that here we have only transport of information from the quantum system to the classical one. We have:

$$\dot{\hat{\rho}} = -i[H, \hat{\rho}] + \kappa(e\hat{\rho}e - \frac{1}{2}\{e, \hat{\rho}\}). \quad (30)$$

The Liouville equation (27) describes time evolution of statistical states of the total system.

that the only measures on the space of pure states of the quantum algebra will be the Dirac measures.

²⁹Thanks are due to N. Gisin for pointing out these references.

³⁰Or, more precisely, as Frobenius–Perron operators. Cf. Ref.[26] for definition and examples of Frobenius–Perron and dual to them Koopman operators.

Let us describe now the PD process associated to this equation. Let T_t be a one-parameter semigroup of (non-linear) transformations of rays in \mathbb{C}^2 given by

$$T(t)\phi = \frac{\phi(t)}{\|\phi(t)\|}, \quad (31)$$

where

$$\phi(t) = \exp\left(-iHt - \frac{\kappa}{2}et\right)\phi. \quad (32)$$

Suppose we start with the quantum system in a pure state ϕ_0 , and the classical system in a state s_1 (resp. s_2). Then ϕ_0 starts to evolve according to the deterministic (but non-linear Schrödinger) evolution $T(t)\phi_0$ until a jump occurs at time t_1 . The time t_1 of the jump is governed by an inhomogeneous Poisson process with the rate function $\lambda(t) = \kappa\|eT(t)\phi_0\|^2$. Classical system switches from s_1 to s_2 (resp. from s_2 to s_1), while $T(t_1)\phi_0$ jumps to $\phi_1 = eT(t_1)\phi_0/\|eT(t_1)\phi_0\|$, and the process starts again. With the initial state being an eigenstate of e , $e\phi_0 = \phi_0$, the rate function λ is approximately constant and equal to κ . Thus $1/\kappa$ can be interpreted as the expected time interval between the successive jumps.

More details about this model illustrating the quantum Zeno effect can be found in Ref. [8].

3.5.2 Simultaneous "Measurement" of Several Noncommuting Observables

Using somewhat pictorial language we can say that in the previous example each *actualization* of the property e was causing a flip in the classical system. In the present example, which is a non-commutative and fuzzy generalization of the model discussed in [7], we consider n , in general *fuzzy*, properties $a_i = a_i^*$, $i = 1, \dots, n$. The Hilbert space \mathcal{H}_q can be completely arbitrary, for instance 2-dimensional. We will denote $a_0^2 \doteq \sum_{i=1}^n a_i^2$. The a_i -s need not be projections, and the different a_i -s need not to commute. The classical system is assumed to have $n+1$ states s_0, s_1, \dots, s_n with s_0 thought of as an initial, neutral state. To each actualization of the property a_i there will be associated a flip between s_0 and s_i . Otherwise the state of the classical system will be unchanged. To this end we take

$$V_1 = \sqrt{\kappa} \begin{pmatrix} 0, & a_1, & 0, & \dots, & 0, & 0 \\ a_1, & 0, & 0, & \dots, & 0, & 0 \\ 0, & 0, & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots, & 0, & 0 \end{pmatrix},$$

$$V_2 = \sqrt{\kappa} \begin{pmatrix} 0, & 0, & a_2, & \dots, & 0, & 0 \\ 0, & 0, & 0, & \dots, & 0, & 0 \\ a_2, & 0, & 0, & \dots, & 0, & 0 \\ 0, & 0, & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots, & 0, & 0 \end{pmatrix},$$

...

$$V_n = \sqrt{\kappa} \begin{pmatrix} 0, & 0, & 0, & \dots, & 0, & a_n \\ 0, & 0, & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots, & 0, & 0 \\ a_n, & 0, & 0, & \dots, & 0, & 0 \end{pmatrix}.$$

The Liouville equation takes now the following form:

$$\dot{\rho}_0 = -i[H, \rho_0] + \kappa \sum_{i=1}^n a_i \rho_i a_i - \frac{\kappa}{2} \{a_0^2, \rho_0\}, \quad (33)$$

$$\dot{\rho}_i = -i[H, \rho_i] + \kappa a_i \rho_0 a_i - \frac{\kappa}{2} \{a_i^2, \rho_i\}. \quad (34)$$

We will derive the PD process for this example in some more details, so that a general method can be seen. First of all we transpose the Liouville equation so as to get time evolution of observables; we use the formula

$$\sum_{\alpha} Tr(\dot{A}_{\alpha} \rho_{\alpha}) = \sum_{\alpha} Tr(A_{\alpha} \dot{\rho}_{\alpha}). \quad (35)$$

In the particular case at hand the evolution equation for observables looks almost exactly the same as that for states:

$$\dot{A}_0 = i[H, A_0] + \kappa \sum_{i=1}^n a_i A_i a_i - \frac{\kappa}{2} \{a_0^2, A_0\}, \quad (36)$$

$$\dot{A}_i = i[H, A_i] + \kappa a_i A_0 a_i - \frac{\kappa}{2} \{a_i^2, A_i\}. \quad (37)$$

Each observable \mathbf{A} of the total system defines now a function $f_{\mathbf{A}}(\psi, \alpha)$ on the space of pure states of the total system

$$f_{\mathbf{A}}(\psi, \alpha) = (\psi, A_{\alpha} \psi). \quad (38)$$

We have to rewrite the evolution equation for observables in terms of the functions $f_{\mathbf{A}}$. To this end we compute the expressions $(\psi, \dot{A}_\alpha \psi)$. Let us first introduce the Hamiltonian vector field X_H on the manifold of pure states of the total system:

$$(X_H f)(\psi, \alpha) = \frac{d}{dt} f(e^{-iHt} \psi)|_{t=0}. \quad (39)$$

Then the terms $(\psi, i[H, A_\alpha] \psi)$ can be written as $(X_H f_{\mathbf{A}})(\psi, \alpha)$. We also introduce vector field X_D corresponding to non-linear evolution:

$$(X_D f)(\psi, \alpha) = \frac{d}{dt} f \left(\frac{\exp(-\kappa t a_\alpha^2 / 2) \psi}{\|\exp(-\kappa t a_\alpha^2 / 2) \psi\|} \right) |_{t=0}. \quad (40)$$

Then evolution equation for observables can be written in a Davis form:

$$\begin{aligned} \frac{d}{dt} f_{\mathbf{A}}(\psi, \alpha) &= ((X_H + X_D) f)(\psi, \alpha) + \\ &+ \lambda(\psi, \alpha) \sum_{\beta} \int Q(\psi, \alpha; d\phi, \beta) (f_{\mathbf{A}}(\phi, \beta) - f_{\mathbf{A}}(\psi, \alpha)), \end{aligned} \quad (41)$$

where Q is a matrix of measures, whose non-zero entries are:

$$Q(\psi, 0; d\phi, i) = \frac{\|a_i \psi\|^2}{\|a_0 \psi\|^2} \delta \left(\phi - \frac{a_i \psi}{\|a_i \psi\|} \right) d\phi, \quad (42)$$

$$Q(\psi, i; d\phi, 0) = \delta \left(\phi - \frac{a_i \psi}{\|a_i \psi\|} \right) d\phi, \quad (43)$$

while

$$\lambda(\psi, \alpha) = \kappa \|a_\alpha \psi\|^2. \quad (44)$$

The symbol $\delta(\phi - \psi) d\phi$ denotes here the Dirac measure concentrated at ψ . We describe now PD process associated to the above semigroup. There are n one-parameter (non-linear) semigroups $T_\alpha(s)$ acting on the space of pure states of the quantum system via

$$\psi \mapsto T_\alpha(t) \psi = \frac{W_\alpha(t) \psi}{\|W_\alpha(t) \psi\|},$$

where

$$W_\alpha(t) = \exp[-iHt - \frac{\kappa}{2} a_\alpha^2 t].$$

If initially the classical system is in a pure state α , and quantum system in a pure state ψ , then quantum system evolves deterministically according to the semigroup T_α : $\psi(t) \mapsto T_\alpha(t) \psi$. The classical system then jumps at the time instant t_1 , determined by the inhomogeneous Poisson process with rate function $\lambda_\alpha = \lambda(\psi, \alpha)$. If the classical system was in one of the states $j = 1, 2, \dots, n$,

then it jumps to 0 with probability one, the quantum state jumps at the same time to the state $a_j\psi(t_1)/\|a_j\psi(t_1)\|$. If, on the other hand, it was in the state 0, then it jumps to one of the states j with probability $\|a_j\psi(t_1)\|^2/\|a_0\psi(t_1)\|^2$. The quantum state jumps at the same time to $a_j\psi(t_1)/\|a_j\psi(t_1)\|$. Let

$$F_\alpha(t) = \exp\left[-\int_0^t \lambda_\alpha(T_\alpha(s)\psi)ds\right].$$

Then F_α is the distribution of t_1 - the first jump time. More precisely, $F_\alpha(t)$ is the survival function for the state α :

$$F_\alpha(t) = P[t_1 > t].$$

Thus the probability distribution of the jump is $p(t) = -dF_\alpha(t)/dt$, and the expected jump time is $\int_0^{+\infty} tp(t)dt$. The probability that the jump will occur between t and $t + dt$, *provided it did not occur yet*, is equal to $1 - \exp\left(-\int_t^{t+dt} \lambda_\alpha(s)ds\right) \approx \lambda_\alpha(t)dt$. Notice that this depends on the actual state (ψ, α) . However, as numerical computation show, the dependence is negligible and approximately jumps occur always after time $t_1 = 1/\kappa$.³¹

3.5.3 Coupling to All One-Dimensional Projections

In the previous example the coupling between classical and quantum systems involved a finite set of non-commuting observables. In the present one we will go to the extreme - we will use all one-dimensional projections in the coupling. One can naturally discover such a model when looking for a precise answer to the question:

how to determine state of an individual quantum system ?

For some time I was sharing the predominant opinion that a positive answer to this question can not be given, as there is no observable to be measured that answers the question: *what state our system is in ?*. Recently Aharonov and Vaidman [1] discussed this problem in some details.³² The difficulty here is in the fact that we have to discriminate between non-orthogonal projections (because *different* states are not necessarily *orthogonal*), and this implies necessity of simultaneous measuring of non-commuting observables. There have been many papers discussing such measurements, different authors taking often different positions. However they all seem to agree on the fact that *predictions* from such measurements are necessarily *fuzzy*. This fuzziness being directly related to the Heisenberg uncertainty relation for non-commuting observables.

³¹This sequence of transformation on the space of pure states of the quantum system can be thought of as a nonlinear version of Barnsley's Iterated Function System (cf. e. g. [4])

³²I do not think that they found the answer, as their arguments are circular, and they seem to be well aware of this circularity.

Using methods and ideas presented in the previous sections of this chapter it is possible to build models corresponding to the intuitive idea of a simultaneous measurement of several non-commuting observables, like, for instance, different spin components, positions and momenta etc. A simple example of such a model was given in the previous section. After playing for a while with similar models it is natural to think of a coupling between a quantum system and a classical device that will result in a determination of the quantum state by the classical device. Ideally, after the interaction, the classical "pointer" should point at some vector in a model Hilbert space. This vector should represent (perhaps, with some uncertainty) the actual state of the quantum system. The model that came out of this simple idea, and which we will now discuss, does not achieve this goal. But it is instructive, as it shows that models of this kind are possible. I believe that one day somebody will invent a better model, a model that can be proven to be optimal, giving the best determination with the least disturbance. Then we will learn something important about the nature of quantum states.

Our model will be formulated for a 2-state quantum system. It is rather straightforward to rewrite it for an arbitrary n -state system, but for $n = 2$ we can be helped by our visual imagination. Thus we take $\mathcal{H}_q = \mathbb{C}^2$ for the Hilbert space of our quantum system. We can think of it as pure spin 1/2. Pure states of the system form up the manifold $\mathcal{S}_q \equiv \mathbb{C}P^1$ which is isomorphic to the 2-sphere $S^2 = \{\mathbf{n} \in \mathbf{R}^3 : \mathbf{n}^2 = 1\}$. Let $\sigma = \{\sigma_i\}$, $i = 1, 2, 3$ denote the Pauli σ -matrices. Then for each $\mathbf{n} \in S^2$ the operator $\sigma(\mathbf{n}) = \sigma \cdot \mathbf{n}$ has eigenvalues $\{+1, -1\}$. We denote by $e(\mathbf{n}) = (I + \sigma(\mathbf{n}))/2$ the projection onto the $+1$ -eigenspace. For the space \mathcal{S}_{cl} of pure states of the classical system we take also S^2 – a copy of \mathcal{S}_q . Notice that S^2 is a homogeneous space for $U(2)$. Let μ be the $U(2)$ invariant measure on S^2 normalized to $\mu(S^2) = 1$. In spherical coordinates we have $d\mu = \sin(\theta)d\phi d\theta/4\pi$. We denote $\mathcal{H}_{tot} = L^2(\mathcal{S}_{cl}, \mathcal{H}_q, d\mu)$ the Hilbert space of the total system, and by $\mathcal{A}_{tot} = L^\infty(\mathcal{S}_{cl}, \mathcal{L}(\mathcal{H}_q), d\mu)$ its von Neumann algebra of observables. Normal states of \mathcal{A}_{tot} are of the form

$$\rho : \mathbf{A} \mapsto \int Tr(A(\mathbf{n})\rho(\mathbf{n}))d\mu(\mathbf{n}),$$

where $\rho \in L^\infty(\mathcal{S}_{cl}, \mathcal{L}(\mathcal{H}_q), d\mu)$ satisfies

$$\begin{aligned} \rho(\mathbf{n}) &\geq 0, \mathbf{n} \in \mathcal{S}_{cl}, \\ \int Tr(\rho(\mathbf{n}))d\mu(\mathbf{n}) &= 1. \end{aligned}$$

We proceed now to define the coupling of the two systems. There will be two constants:

- κ – regulating the time rate of jumps
- ω – entering the quantum Hamiltonian

The idea is that if the quantum system is at some pure state \mathbf{n}_q , and if the classical system is in some pure states \mathbf{n}_{cl} , then \mathbf{n}_{cl} will cause the Hamiltonian rotation of \mathbf{n}_q around \mathbf{n}_{cl} with frequency ω , while \mathbf{n}_q will cause, after a random waiting time t_1 proportional to $1/\kappa$, a jump, along geodesics, to the "other side" of \mathbf{n}_q . The classical transformation involved is nothing but a geodesic symmetry on the symmetric space $\mathbb{C}P^2 = U(2)/(U(1) \times U(1))$. It has the advantage that it is a measure preserving transformation. It has a disadvantage because \mathbf{n}_{cl} *overjumps* \mathbf{n}_q .

We will use the notation $\mathbf{n}(\mathbf{n}')$ to denote the π rotation of \mathbf{n}' around \mathbf{n} . Explicitly:

$$\mathbf{n}(\mathbf{n}') = 2(\mathbf{n} \cdot \mathbf{n}')\mathbf{n} - \mathbf{n}'.$$

For each \mathbf{n} we define $V_{\mathbf{n}} \in \mathcal{L}(\mathcal{H}_{tot})$ by

$$(V_{\mathbf{n}}\Psi)(\mathbf{n}') = \sqrt{\kappa}e(\mathbf{n})\Psi(\mathbf{n}(\mathbf{n}')). \quad (45)$$

Using $V_{\mathbf{n}}$ -s we can define Lindblad-type coupling between the quantum system and the classical one. To give our model more flavor, we will introduce also a quantum Hamiltonian that depends on the actual state of the classical system; thus we define

$$(H\Psi)(\mathbf{n}) = H(\mathbf{n})\Psi(\mathbf{n}) = \frac{\omega}{2}\sigma(\mathbf{n})\Psi(\mathbf{n}). \quad (46)$$

Our coupling is now given by

$$\mathcal{L}_{cq}\rho = -i[H, \rho] + \int_{\mathcal{S}_{cl}} \left(V_{\mathbf{n}}\rho V_{\mathbf{n}}^* - \frac{1}{2}\{V_{\mathbf{n}}^*V_{\mathbf{n}}, \rho\} \right) d\mu(\mathbf{n}). \quad (47)$$

Notice that $V_{\mathbf{n}}^* = V_{\mathbf{n}}$ and $V_{\mathbf{n}}^2 = \kappa e(\mathbf{n})$. Now, $\int e(\mathbf{n})d\mu(\mathbf{n})$ being $U(2)$ -invariant, it must be proportional to the identity. Taking its trace we find that

$$\int e(\mathbf{n})d\mu(\mathbf{n}) = I/2,$$

and therefore

$$\mathcal{L}_{cq}\rho = -i[H, \rho] + \int V_{\mathbf{n}}\rho V_{\mathbf{n}} d\mu(\mathbf{n}) - \frac{\kappa}{2}\rho. \quad (48)$$

Explicitly, using the definition of $V_{\mathbf{n}}$, we have

$$(\mathcal{L}_{cq}\rho)(\mathbf{n}) = -i\frac{\omega}{2}[\sigma(\mathbf{n}), \rho(\mathbf{n})] + \kappa \int e(\mathbf{n}')\rho(\mathbf{n}'(\mathbf{n}))e(\mathbf{n}')d\mu(\mathbf{n}') - \frac{\kappa}{2}\rho(\mathbf{n}). \quad (49)$$

Notice that for each operator $a \in \mathcal{L}(\mathcal{H}_q)$ we have the following formula: ³³

$$\int e(\mathbf{n})a e(\mathbf{n})d\mu(\mathbf{n}) = \frac{1}{6}(a + Tr(a)I). \quad (50)$$

³³The formula is easily established for a of the form $e(\mathbf{n}')$, and then extended to arbitrary operators by linearity.

If $\omega = 0$, that is if we neglect the Hamiltonian part, then using this formula we can integrate over \mathbf{n}' to get the effective Liouville operator for the quantum state $\hat{\rho} = \int \rho(\mathbf{n}) d\mu(\mathbf{n})$:

$$\mathcal{L}_{cq}\hat{\rho} = \frac{\kappa}{6} (I - 2\hat{\rho}), \quad (51)$$

with the solution

$$\hat{\rho}(t) = \exp(-\kappa t/3) \rho(0) + \frac{1 - \exp(-\kappa t/3)}{2} I. \quad (52)$$

It follows that, as the result of the coupling, the effective quantum state undergoes a rather uninteresting time-evolution: it dissipates exponentially towards the totally mixed state $\frac{I}{2}$, and this does not depend on the initial state of the classical system.

Returning back to the case of non-zero ω we discuss now the piecewise deterministic random process of the two pure states \mathbf{n}_q and \mathbf{n}_{cl} . To compute it we proceed as in the previous example, with the only change that now pure states of the quantum *and* of the classical system are parameterized by the same set - S^2 in our case. To keep track of the origin of each parameter we will use subscripts as in \mathbf{n}_{cl} and \mathbf{n}_q . As in the previous example each observable \mathbf{A} of the total system determines a function $f_{\mathbf{A}} : S^2 \times S^2 \rightarrow \mathbb{C}$ by

$$f_{\mathbf{A}}(\mathbf{n}_q, \mathbf{n}_{cl}) = Tr(e(\mathbf{n}_q) A(\mathbf{n}_{cl})).$$

The Liouville operator \mathcal{L}_{cq} , acting on observables, can be then rewritten in terms of the functions $f_{\mathbf{A}}$:

$$\begin{aligned} (\mathcal{L}_{cq} f_{\mathbf{A}})(\mathbf{n}_q, \mathbf{n}_{cl}) &= (X_H f)(\mathbf{n}_q, \mathbf{n}_{cl}) + \\ &+ \kappa \int p(\mathbf{n}_q, \mathbf{n}'_q) f_{\mathbf{A}}(\mathbf{n}'_q, \mathbf{n}'_q(\mathbf{n}_q)) d\mu(\mathbf{n}'_q) + \\ &- \frac{\kappa}{2} f_{\mathbf{A}}(\mathbf{n}_q, \mathbf{n}_{cl}), \end{aligned} \quad (53)$$

where X_H is the Hamiltonian vector field

$$(X_H f)(\mathbf{n}_q, \mathbf{n}_{cl}) = \left. \frac{d}{dt} f\left(e^{-iH(\mathbf{n}_{cl})t} \cdot \mathbf{n}_q\right) \right|_{t=0}, \quad (54)$$

and

$$p(\mathbf{n}, \mathbf{n}') = Tr(e(\mathbf{n}) e(\mathbf{n}')) = (1 + \mathbf{n} \cdot \mathbf{n}')/2 \quad (55)$$

is known as the transition probability between the two quantum states. The PD process on $\mathcal{S}_q \times \mathcal{S}_{cl}$ can now be described as follows. Let $\mathbf{n}_q(0)$ and $\mathbf{n}_{cl}(0)$ be the initial states of the quantum and of the classical system. Then the quantum system evolves unitarily according to the quantum Hamiltonian

$H(\mathbf{n}_{cl})$ until at a time instant t_1 a jump occurs. The time rate of jumps is governed by the homogeneous Poisson process with rate $\kappa/2$. The quantum state $\mathbf{n}_q(t_1)$ jumps to a new state \mathbf{n}'_q with probability distribution $p(\mathbf{n}_q(t_1), \mathbf{n}'_q)$ while \mathbf{n}_{cl} jumps to $\mathbf{n}'_q(\mathbf{n}_q(t_1))$ and the process starts again (see Fig. 2).

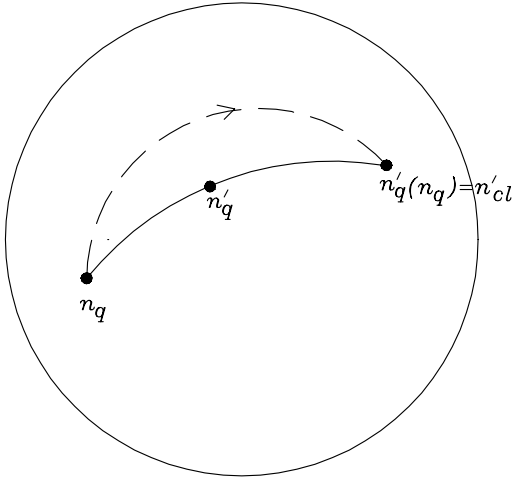


Figure 2: The quantum state $\mathbf{n}_q(t_1)$ jumps to a new state \mathbf{n}'_q with probability distribution $p(\mathbf{n}_q(t_1), \mathbf{n}'_q)$ while \mathbf{n}_{cl} jumps to $\mathbf{n}'(\mathbf{n}_q(t_1))$.

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